Phys 570A Midterm 2 Solutions

1) a) Momentum representation:

$$\vec{r} = i\hbar \frac{\partial}{\partial \vec{p}}$$

$$[\vec{r}, \hat{T}(\vec{a})]\tilde{\Psi}(\vec{p}) = i\hbar\left(\frac{\partial}{\partial \vec{p}}\left[\hat{T}(\vec{a})\tilde{\Psi}(\vec{p})\right] - \hat{T}(\vec{a})\frac{\partial \tilde{\Psi}}{\partial \vec{p}}\right)$$

$$= i\hbar \frac{\partial \hat{T}(\vec{a})}{\partial \vec{p}}\tilde{\Psi}(\vec{p})$$

$$\Rightarrow [\vec{r}, \hat{T}(\vec{a})] = i\hbar \frac{\partial \hat{T}(\vec{a})}{\partial \vec{p}} = -\vec{a}\,\hat{T}(\vec{a})$$

b) $\vec{r}\,\hat{T}(\vec{a})|\vec{r}_0\rangle = \hat{T}(\vec{a})\vec{r}|\vec{r}_0\rangle - \vec{a}\,\hat{T}(\vec{a})|\vec{r}_0\rangle$

$$= (\vec{r}_0 - \vec{a})\,\hat{T}(\vec{a})|\vec{r}_0\rangle$$

Position eigenstate with eigenvalue

$\vec{r}_0 - \vec{a}$.

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2) a) $i\hbar \frac{da}{dt} = [a, H] = \hbar\omega[a, a^\dagger a]$

$$= \hbar\omega[a, a^\dagger]a = \hbar\omega a$$

$$\frac{da}{dt} = -i\omega a \Rightarrow \boxed{a(t) = a(0)e^{-i\omega t}}$$

$$i\hbar\frac{da^\dagger}{dt} = [a^\dagger, H] = \hbar\omega[a^\dagger, a^\dagger a] = \hbar\omega a^\dagger[a^\dagger, a]$$

$$= -\hbar\omega a^\dagger$$

$$\frac{da^\dagger}{dt} = i\omega a^\dagger \Rightarrow \boxed{a^\dagger(t) = a^\dagger(0)e^{i\omega t}}$$

2) b) $\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a^{+}_{(t)} + a(t)\right)$

$$\hat{P}_x(t) = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^{+}_{(t)} - a(t)\right)$$

$$\langle x(t)\rangle = \left(\alpha^*\langle 0| + \beta^*\langle 1|\right) a^{+}(t) + a(t)\left(\alpha|0\rangle + \beta|1\rangle\right)\sqrt{\frac{\hbar}{2m\omega}}$$

$$= \left(\beta^*\alpha\langle 1|a^{+}e^{i\omega t}|0\rangle + \alpha^*\beta\langle 0|a e^{-i\omega t}|1\rangle\right)\sqrt{\frac{\hbar}{2m\omega}}$$

$$= \left(\beta^*\alpha e^{i\omega t} + \alpha^*\beta e^{-i\omega t}\right)\sqrt{\frac{\hbar}{2m\omega}}$$

Let $\alpha = |\alpha| e^{i\delta}$, $\beta = |\beta| e^{i\gamma}$, $\delta - \gamma = \theta$

$$\boxed{\langle x(t)\rangle = \sqrt{\frac{2\hbar}{m\omega}}\cos(\omega t + \theta)|\alpha||\beta|}$$

$$\langle P_x(t)\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\beta^*\alpha e^{i\omega t} - \alpha^*\beta e^{-i\omega t}\right)$$

$$\boxed{\langle P_x(t)\rangle = -\sqrt{2m\hbar\omega}\sin(\omega t + \theta)|\alpha||\beta|}$$

c) $\hat{x}^2(t) = \frac{\hbar}{2m\omega}\left((a^{+}_{(t)})^2 + (a(t))^2 + a^{+}_{(t)}a(t) + a(t)a^{+}(t)\right)$

$$a(t)a^{+}(t) = a^{+}(t)a(t) + 1 \qquad [a(t), a^{+}(t)] = 1$$

$$\langle\psi|\hat{x}^2(t)|\psi\rangle = \frac{\hbar}{2m\omega}\left(|\alpha|^2\langle 0|2a^{+}a + 1|0\rangle + |\beta|^2\langle 1|2a^{+}a + 1|1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\left(|\alpha|^2 + 3|\beta|^2\right) = \langle x^2(t)\rangle$$

$$\hat{P}_x^2(t) = -\frac{m\hbar\omega}{2}\left((a^{+}(t))^2 + (a(t))^2 - a^{+}(t)a(t) - a(t)a^{+}(t)\right)$$

$$\langle\psi|\,\hat{p}_x^2(t)\,|\psi\rangle = \frac{m\hbar\omega}{2}\langle\psi|\,2a^\dagger a+1\,|\psi\rangle$$

$$= \frac{m\hbar\omega}{2}\left(|\alpha|^2 + 3|\beta|^2\right)$$

$$\langle\Delta x^2(t)\rangle = \langle x^2(t)\rangle - \langle x(t)\rangle^2$$

$$= \frac{\hbar}{2m\omega}\left(|\alpha|^2+3|\beta|^2-4|\alpha|^2|\beta|^2\cos^2(\omega t+\theta)\right) > 0$$

$$\langle\Delta p_x^2(t)\rangle = \langle p_x^2(t)\rangle - \langle p_x(t)\rangle^2$$

$$= \frac{m\hbar\omega}{2}\left(|\alpha|^2+3|\beta|^2-4|\alpha|^2|\beta|^2\sin^2(\omega t+\theta)\right)$$

$$> 0$$

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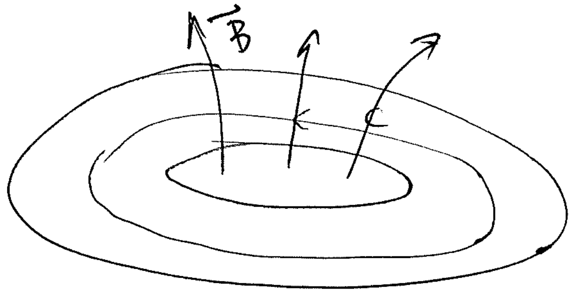
3) a) $\vec{v} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$, $\vec{p} = \frac{\hbar}{i}\nabla$

$$\vec{J}_e = \frac{q}{m}\,\mathrm{Re}\left\{\sqrt{n_s}\,e^{-i\theta}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)\sqrt{n_s}\,e^{i\theta(\vec{r})}\right\}$$

$$= \frac{n_s q}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A}\right)$$

$$\nabla\times\vec{J}_e = -\frac{n_s q^2}{mc}\nabla\times\vec{A} = -\frac{n_s q^2}{mc}\vec{B}$$ ✓

b)



$\vec{J}_e = 0$ and $\vec{B} = 0$ along C.

$$0 = \frac{n_s q}{m}\left(\hbar \nabla\theta - \frac{q}{c}\vec{A}\right) \quad \text{along } C$$

$$\nabla\theta = \frac{q}{\hbar c}\vec{A} \quad \text{along } C$$

$$\oint_C \nabla\theta \cdot \vec{dl} = \frac{q}{\hbar c}\oint_C \vec{A}\cdot d\vec{l} = \frac{q}{\hbar c}\Phi$$

$$\shortparallel$$

$2\pi s$, $s \in \mathbb{Z}$ because ψ is single-valued.

$$2\pi s = \frac{q}{\hbar c}\Phi$$

$$\Phi = \frac{2\pi\hbar c}{q} s = \frac{hc}{2e} s = \phi_0^{sc} s$$

$$\phi_0^{sc} = \frac{hc}{2e}$$