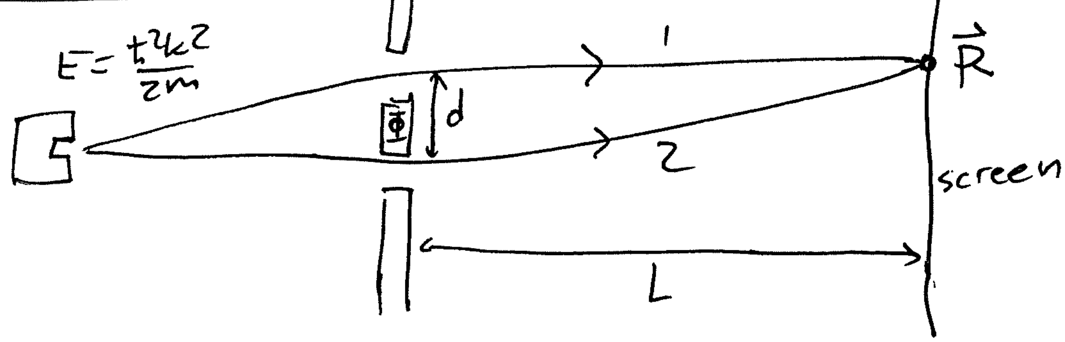
Practice problem solutions

1)



$$E\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)^2 \psi$$ (V = 0 in region of interest)

$\vec{B} = 0$ wherever $\psi \neq 0$, so we can set

$\vec{A} = \nabla f(\vec{r}) = \frac{\hbar c}{q}\nabla\theta$ in Sch. eq.

Let $\psi = e^{i\theta(\vec{r})}\psi'$. ψ' satisfies

$$E\psi' = -\frac{\hbar^2}{2m}\nabla^2\psi'.$$

Interference pattern:

$$\rho(\vec{R}) = |\psi_1(\vec{R}) + \psi_2(R)|^2 = |\psi_1'(\vec{R})e^{i\theta_1} + \psi_2'(\vec{R})e^{i\theta_2}|^2$$

$\psi_1'(\vec{R}) = \sqrt{\rho_1}\,e^{ikL_1}$, $\psi_2'(\vec{R}) = \sqrt{\rho_2}\,e^{ikL_2}$,

where ρ_1 and ρ_2 are the intensities on the screen with only slit 1 or 2 open. $\Delta L = L_2 - L_1 = d\sin\theta$

$$\rho(\vec{R}) = \rho_1 + \rho_2 + 2\sqrt{\rho_1\rho_2}\cos(k\Delta L + \theta_2 - \theta_1)$$

$$\theta_1 = \int_1 \frac{e}{\hbar c}\vec{A}\cdot d\vec{l}\,, \quad \theta_2 = \int_2 \frac{e}{\hbar c}\vec{A}\cdot d\vec{l}$$

$$\theta_2 - \theta_1 = \frac{e}{\hbar c}\oint \vec{A}\cdot d\vec{l} = \frac{e}{\hbar c}\Phi = \frac{2\pi\Phi}{\phi_0}$$

$$\phi_0 = \frac{hc}{e} = \text{flux quantum.}$$

<u>Constructive interference</u> (bright fringe)

occurs when

$$k\Delta L + \frac{2\pi\Phi}{\phi_0} = 2\pi n\,, \qquad n \in \mathbb{Z}$$

$$\frac{2\pi\, d\sin\theta}{\lambda} + \frac{2\pi\Phi}{\phi_0} = 2\pi n$$

$$\boxed{\frac{d\sin\theta}{\lambda} = n - \frac{\Phi}{\phi_0}}$$

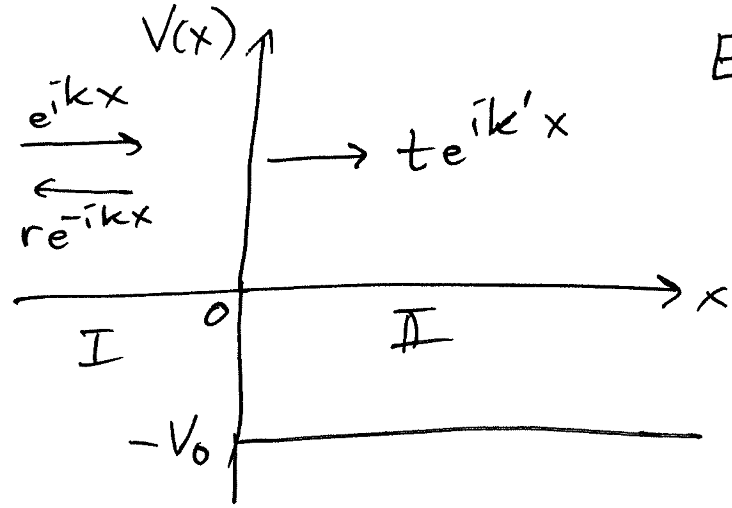
<u>Destructive interference</u> (dark fringe)

occurs when

$$k\Delta L + \frac{2\pi\Phi}{\phi_0} = 2\pi\left(n + \tfrac{1}{2}\right)$$

$$\frac{d\sin\theta}{\lambda} = n + \frac{1}{2} - \frac{\Phi}{\phi_0}$$

2) a)



$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k'^2}{2m} - V_0$$

$$\psi_I(0) = \psi_{II}(0) \qquad \psi_I'(0) = \psi_{II}'(0)$$

$$1 + r = t \quad (1) \qquad ik(1-r) = ik' t$$

$$1 - r = \frac{k'}{k} t \quad (2)$$

Add (1) + (2):

$$2 = \left(1 + \frac{k'}{k}\right) t = \frac{k+k'}{k} t$$

$$t = \frac{2k}{k+k'}$$

$$j_{in} = \frac{1}{m} \mathrm{Re}\left(e^{-ikx} \frac{\hbar}{i} \frac{d}{dx} e^{ikx}\right) = \frac{\hbar k}{m}$$

$$j_{tr} = \frac{1}{m} \mathrm{Re}\left(t^* e^{-ik'x} \frac{\hbar}{i} \frac{d}{dx} t e^{ik'x}\right) = |t|^2 \frac{\hbar k'}{m}$$

<u>Transmission probability</u>

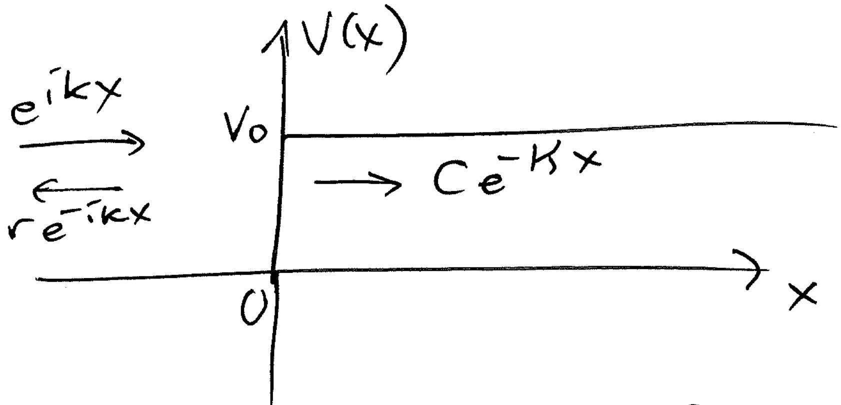
$$T = \frac{|j_{tr}|}{|j_{in}|} = \frac{k'}{k} |t|^2 = \frac{4kk'}{(k+k')^2}$$

If $E = \frac{V_0}{99}$, then $\frac{\hbar^2 k'^2}{2m} = E + V_0 = 100E = 100 \frac{\hbar^2 k^2}{2m}$

$$k' = 10k$$

$$T = \frac{40k^2}{(11k)^2} = \frac{40}{121} = 0.33$$

b) Now $V(x) = V_0\,\theta(x)$, $E < V_0$.



$$E = \frac{\hbar^2 k^2}{2m} = V_0 - \frac{\hbar^2 K^2}{2m}, \quad K > 0$$

$$\psi_I(0) = \psi_{II}(0) \qquad \psi_I'(0) = \psi_{II}'(0)$$

$$1 + r = C \quad (3) \qquad ik(1-r) = -KC$$

$$1 - r = i\frac{K}{k}\,C \quad (4)$$

Add (3) & (4):

$$2 = \left(1 + i\frac{K}{k}\right)C = \frac{k+iK}{k}\,C$$

$$C = \frac{2k}{k+iK} \qquad \psi_{II}(x) = \frac{2k}{k+iK}\,e^{-Kx}$$

$$j_{tr} = \frac{1}{m}\,\mathrm{Re}\left\{\psi_{II}^*(x)\,\frac{\hbar}{i}\frac{d}{dx}\,\psi_{II}(x)\right\}$$

$$= \frac{1}{m}\,\mathrm{Re}\left\{|C|^2\, i\hbar K\right\} = 0$$

$\therefore$ $T = 0$.

3) a) ψ_1

b) $\psi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$

possible outcomes are b_1 and b_2

$P(b_1) = \frac{1}{2}$, $P(b_2) = \frac{1}{2}$

c) First, solve for ϕ_1 & ϕ_2:

$\sqrt{2}\,\psi_1 = \phi_1 + \phi_2$ $\qquad \phi_1 = \frac{\psi_1 + \psi_2}{\sqrt{2}}$

$\sqrt{2}\,\psi_2 = \phi_1 - \phi_2$ $\qquad \phi_2 = \frac{\psi_1 - \psi_2}{\sqrt{2}}$

If b measurement yields b_1, a subsequent measurement of a will yield a_1 with probability 50%.

If b meas. yields b_2, a subseq. meas. of a will yield a_1 with probability 50%. $\therefore$ The overall probability of getting a_1 again is 50%.

4) a) $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$

Let $|\lambda\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, where

$\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$.

$$\hat{a}|\lambda\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{k=0} c_{k+1}\sqrt{k+1}\,|k\rangle$$

$$= \lambda|\lambda\rangle = \sum_{k=0}^{\infty} \lambda c_k |k\rangle$$

$$\Rightarrow \quad \lambda c_k = c_{k+1}\sqrt{k+1}$$

$$\frac{c_{k+1}}{c_k} = \frac{\lambda}{\sqrt{k+1}}$$

$$\frac{c_1}{c_0} = \lambda, \quad \frac{c_2}{c_1} = \frac{\lambda}{\sqrt{2}}, \quad \frac{c_3}{c_2} = \frac{\lambda}{\sqrt{3}}, \ldots$$

$$\frac{c_2}{c_0} = \frac{\lambda^2}{\sqrt{2!}}, \quad \frac{c_3}{c_0} = \frac{\lambda^3}{\sqrt{3!}}, \ldots$$

$$\frac{c_n}{c_0} = \frac{\lambda^n}{\sqrt{n!}}, \quad |\lambda\rangle = \sum_{n=0}^{\infty} c_0 \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

$$1 = \langle\lambda|\lambda\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} = |c_0|^2 e^{\lambda^2}$$

$$C_0 = e^{-\lambda^2/2} . \qquad |\lambda\rangle = e^{-\lambda^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

Notice that $e^{\lambda a^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{(\lambda a^\dagger)^n}{n!}|0\rangle$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle .$$

$$\therefore \quad |\lambda\rangle = e^{-\lambda^2/2} e^{\lambda a^\dagger} |0\rangle .$$

b) Suppose $\exists\ |\nu\rangle$ s.t. $a^\dagger|\nu\rangle = \nu|\nu\rangle$

Let $|\nu\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$

$$a^\dagger|\nu\rangle = \sum_{n=0}^{\infty} \sqrt{n+1}\, b_n |n+1\rangle = \sum_{k=1}^{\infty} \sqrt{k}\, b_{k-1} |k\rangle$$

$$= \nu|\nu\rangle = \sum_{k=0}^{\infty} \nu b_k |k\rangle$$

$$\Rightarrow \quad \nu b_k = \sqrt{k}\, b_{k-1}$$

$$\nu b_0 = 0$$

$$\nu b_1 = b_0 = 0 , \ldots$$

All coefficients are zero.

$\therefore$ $|\nu\rangle = 0$, not normalizable.

There is no state in Hilbert space (ket) that is an eigenstate of $a^\dagger$.

On the other hand, $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$.
$\exists$ a bra $\langle\lambda|$ that is an eigenstate of $a^\dagger$.

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5) $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|\uparrow\rangle + b|\downarrow\rangle$

a) $S_z = \pm\frac{\hbar}{2}$

$P(S_z = \frac{\hbar}{2}) = |a|^2$, $P(S_z = -\frac{\hbar}{2}) = |b|^2$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(|a|^2 - |b|^2\right)$$

b) $S_y = \pm\frac{\hbar}{2}$ (eigenvalues of $\hat{S}_y = \frac{\hbar}{2}\sigma_y$).

$$\langle S_y \rangle = \frac{\hbar}{2}(a^* \;\; b^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{\hbar}{2}(a^* \;\; b^*)\begin{pmatrix} -ib \\ ia \end{pmatrix} = \frac{\hbar}{2}\left(i a b^* - i a^* b\right)$$

$$= \frac{\hbar}{2} Re\{i a b^*\} = \frac{\hbar}{2}\langle\sigma_y\rangle$$

$$= \frac{\hbar}{2}\left[P(S_y = \tfrac{\hbar}{2}) - P(S_y = -\tfrac{\hbar}{2})\right]$$

$P(S_y = \frac{\hbar}{2}) - P(S_y = -\frac{\hbar}{2}) = \langle\sigma_y\rangle$
$P(S_y = \frac{\hbar}{2}) + P(S_y = -\frac{\hbar}{2}) = 1$

$$P(S_y = \tfrac{\hbar}{2}) = \frac{1 + \langle \sigma_y \rangle}{2} = \frac{1}{2} + \frac{1}{2} Re\{iab^*\}$$

$$P(S_y = -\tfrac{\hbar}{2}) = \frac{1 - \langle \sigma_y \rangle}{2} = \frac{1}{2} - \frac{1}{2} Re\{iab^*\}$$

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6) a) $J_z|\psi\rangle = (m_\ell + m_s)\hbar|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$

a) $J_z = +\frac{\hbar}{2}$ 100% probability.

b) $|\psi\rangle = \alpha|j=\tfrac{3}{2}, m_j = \tfrac{1}{2}\rangle + \beta|j=\tfrac{1}{2}, m_j=\tfrac{1}{2}\rangle$

$$J_+|\psi\rangle = \alpha\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{3}{2})}\,|j=\tfrac{3}{2}, m_j = \tfrac{3}{2}\rangle$$

$$= \alpha\sqrt{3}\,|j=\tfrac{3}{2}, m_j = 3/2\rangle$$

on the other hand,

$$J_+|\psi\rangle = (L_+ + S_+)|\psi\rangle = \hbar|\ell=1, m_\ell=1\rangle|\uparrow\rangle$$

$$= \hbar|j=\tfrac{3}{2}, m_j = \tfrac{3}{2}\rangle$$

$\Rightarrow \quad \alpha = \sqrt{\tfrac{1}{3}} \qquad |\beta| = \sqrt{\tfrac{2}{3}}$

$\vec{J}^2 = \hbar^2 j(j+1) \qquad P(j=\tfrac{3}{2}) = \tfrac{1}{3}$

$P(j=\tfrac{1}{2}) = \tfrac{2}{3}$

7) $|\psi_f\rangle = \sqrt{\frac{1}{2}}\left(e^{i\theta}|\uparrow\downarrow\rangle + e^{-i\theta}|\downarrow\uparrow\rangle\right)$

$|\psi_f\rangle$ is a linear combination of

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\vec{S}^2 = \hbar^2 s(s+1), \quad s = 0, 1$$

$$P(s=0) = |\langle 00|\psi_f\rangle|^2 = \sin^2\theta$$

$$P(s=1) = |\langle 10|\psi_f\rangle|^2 = \cos^2\theta$$

$$\langle 00|\psi_f\rangle = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i\sin\theta$$

$$\langle 10|\psi_f\rangle = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta$$