How to antisymmetrize scattering states?

This is the attempt to understand the application of the Pauli principle on scattering states. The ideas are from Goldberger-Watson. The main subject is to rederive the scattering cross sections of Glöckles report on 3N scattering.

Antisymmetrizer

For simplicity I will deal only with one kind of fermions. This should be easily generalized to bosons or several kinds of particles.

The antisymmetrizer is a sum over all permutations P of N particles

$$\mathcal{S} = \frac{1}{N!} \sum_P \epsilon_P \; P \tag{1}$$

This is normalized to make $\mathcal{S}$ an hermitian projection operator.

$$\begin{aligned} &\mathcal{S}^2 = \mathcal{S} \\ &< \mathcal{S}a|\mathcal{S}b >=< a|\mathcal{S}^2 b >=< a|\mathcal{S}b > \end{aligned} \tag{2}$$

Channels states

The non-antisymmetrized channel states describe motion of particles well separated from each other (no overlaps) or clusters of such particles (composite particles without overlap).

$$|a >= |(1...v_1)(v_1...v_1 + v_2)... > \tag{3}$$

The first cluster comprises v_1, the second one v_2, ... particles. Each subgroup of particles is in an antisymmetrized bound state of v_i particles. Therefore, the norms of these channel states, when a permutation P is applied, are

$$\begin{aligned} &\epsilon_P < a|P|a >=< (1...v_1)(v_1...v_1 + v_2)...|(P_1...P_{v_1})(P_{v_1}...P_{v_1+v_2})... > \\ &= \left\{ \begin{array}{ll} 1 & \text{for } P \text{ exchanges particles only within clusters} \\ 0 & \text{for } P \text{ exchanges particles between clusters} \end{array} \right. \end{aligned} \tag{4}$$

This means that the normalized antisymmetrized state reads

$$\{|a >\}_A = C\mathcal{S} \; |a >= \frac{1}{\sqrt{N!v_1!v_2!...}} \; \sum_P \epsilon_P \; P \; |a > \tag{5}$$

and that $C = \sqrt{\frac{N!}{v_1!v_2!...}}$.

Equations for scattering states

Here we summarize very symbolically the definitions of scattering states. Mostly to establish that the antisymmetrized scattering state is based on the antisymmetrized channel state.

$$|a >^{\pm} = \frac{\pm i\epsilon}{E \pm i\epsilon - H} \; |a > \tag{6}$$

Of course the antisymmetrizer commutes with the Hamiltonian. Therefore

$$\{|a>^{\pm}\}_A = C\mathcal{S}\ |a>^{\pm} = \frac{\pm i\epsilon}{E \pm i\epsilon - H}\ \{|a>\}_A \tag{7}$$

Because of this relation, one can set up a set of LS equations for each channel state $P|a>^{\pm}$ and add up the solutions to the completely antisymmetrized states $\{|a>^{\pm}\}_A$. In the following derivation of the scattering transition operator, I assume that I have the same antisymmetrizer on the left and right hand side. This is the case in our pion production, because the number of nucleons is unchanged.

I decompose the Hamiltonian as $H = H_a + V^a$. This only works for the non-antisymmetrized channel state. The channel state is an eigen state of H_a

$$H_a\ |a> = E_a\ |a> \tag{8}$$

and V^a are the interactions between the clusters of $|a>$. With the resolvent's identity, one gets

$$\frac{\pm i\epsilon}{E \pm i\epsilon - H}\ |a> = |a> + \frac{1}{E \pm i\epsilon - H}\ V^a\ \frac{\pm i\epsilon}{E \pm i\epsilon - H_a}\ |a> = \frac{1}{E \pm i\epsilon - H}\ V^a\ |a> \tag{9}$$

and this leads to a relation between $\pm$ states

$$\{|a>^-\}_A - \{|a>^+\}_A = 2\pi i\delta(E - H)C\mathcal{S}\ V^a\ |a> \tag{10}$$

This relation introduces the transition operator in the S-matrix

$$S_{ab} = \{^-< a|\}_A|\{|b>^+\}_A = \delta_{ab} + 2\pi i\delta(E_a - E_b) < a|C_a\ V^a\ \mathcal{S}\ \{|b>^+\}_A \tag{11}$$

The δ function is the result of the orthogonality of scattering states. So we have to look at the matrix elements $C_a < a|\ V^a\ \mathcal{S}\ \{|b>^+\}_A$ in our case. The incoming state is an eigenstate of the antisymmetrizer with eigenvalue one. Therefore, we can drop the $\mathcal{S}$ operator and end up with

$$M_{ab} = C_a < a|\ V^a\ \{|b>^+\}_A \tag{12}$$

3N system as example

The main object of this section is to establish that all considerations work also for the known case of 3N scattering. I look at two cases: a) elastic Nd scattering and b) break up scattering.

a) For the incoming state $|b>$ the normalization constant is $C_b = \sqrt{3!/2!}$ as shown above. The state reads

$$\{|b>^+\}_A = \frac{1}{\sqrt{3}}\ (|(12)3>^+ + |(23)1>^+ + |(31)2>^+) \tag{13}$$

The same C_a applies to the incoming state. This means the matrix element reads

$$M_{ab} = \sqrt{3}\ \frac{1}{\sqrt{3}}\ < (12)3|V^a(|(12)3>^+ + |(23)1>^+ + |(31)2>^+) \tag{14}$$

This is in agreement with Glöckle's result of the report on 3N scattering.

b) For the incoming state $|b>$ the normalization constant is $C_b = \sqrt{3!/2!}$ as in a). The state reads

$$\{|b>^+\}_A = \frac{1}{\sqrt{3}} (|(12)3>^+ + |(23)1>^+ + |(31)2>^+) \tag{15}$$

For the 3N breakup state there are no clusters. Therefore $C_a = \sqrt{6}$ and

$$M_{ab} = \sqrt{6}\,\frac{1}{\sqrt{3}} \; < 123|V^a(|(12)3>^+ + |(23)1>^+ + |(31)2>^+) \tag{16}$$

The outgoing state is not antisymmetrized in the pair (12). This remains to be done and leads to

$$\sqrt{2} \; < 123| = < 123|_A + < 123|_S \tag{17}$$

Here the indices indicate the symmetrization of the pair (12). For breakup $V^a(|(12)3>^+ + |(23)1>^+ + |(31)2>^+)$ is antisymmetric in the pair (12) therefore the symmetric part does not contribute and we find

$$M_{ab} = {}_A< 123|V^a(|(12)3>^+ + |(23)1>^+ + |(31)2>^+) \tag{18}$$

This is again in agreement with Glöckle's result of the report on 3N scattering.

Elastic dd scattering

This is something to be compared to Antonio's results. The established scheme gives $C_b = \sqrt{4!/(2!2!)}$. The incoming state reads

$$\begin{aligned} |b>^+{}_A \;&=\; \frac{1}{\sqrt{6}} (|(12)(34)>^+ + |(23)(14)>^+ + |(31)(24)>^+ \\ &\qquad + |(34)(12)>^+ + |(14)(23)>^+ + |(24)(31)>^+) \end{aligned} \tag{19}$$

The matrix element for elastic scattering is then

$$\begin{aligned} M_{ab} \;&=\; < (12)(34)|V^a(|(12)(34)>^+ + |(23)(14)>^+ + |(31)(24)>^+ \\ &\qquad + |(34)(12)>^+ + |(14)(23)>^+ + |(24)(31)>^+) \end{aligned} \tag{20}$$

The $|(12)(34)>$ states are characterized by two momenta p_1 and p_2 of the deuterons. Because the original expression is based on the antisymmetric states in Eq. (11), the M_{ab} matrix automatically accounts for the fact, that one cannot distinguish the deuterons. However, it also means that states, where p_1 and p_2 are exchanged are the same. Therefore integrating over the full space p_1,p_2 leads to a $2! = 2$ fold overcounting. The total cross section is proportional to

$$\sigma \propto \frac{1}{2} \int d^3p_1 d^3p_2 \; \delta^4(P_{tot} - P'_{tot})|M_{ab}|^2 \tag{21}$$

The process dd to $\alpha\ \pi$

Here there is only one cluster in the outgoing state. Therefore $C_a = \sqrt{4!/4!}$. V^a are the interactions of nucleons with the pions. This interaction involves all nucleons and is symmetric in the nucleons. The incoming state is given as above for dd elastic scattering. Therefore

$$M_{ab} \;=\; \sqrt{1}\,\frac{1}{\sqrt{6}} \; < 1234|V^a(|(12)(34)>^+ + |(23)(14)>^+ + |(31)(24)>^+$$

$$
\begin{aligned}
& +|(34)(12)>^{+} +|(14)(23)>^{+} +|(24)(31)>^{+}) \\
= \quad & \sqrt{6}\ <1234|V^{a}|(12)(34)>^{+}
\end{aligned}
\tag{22}
$$

As usual the pion states have been omitted. This means Antonio is right, because the incoming current is the same as if calculated with non-antisymmetrized states. For the calculation of the total cross section, there is no overcounting possible, because we only have one final nucleon state, the α bound state. Therefore the $\sqrt{6}$ cannot cancel because of a double counting argument.