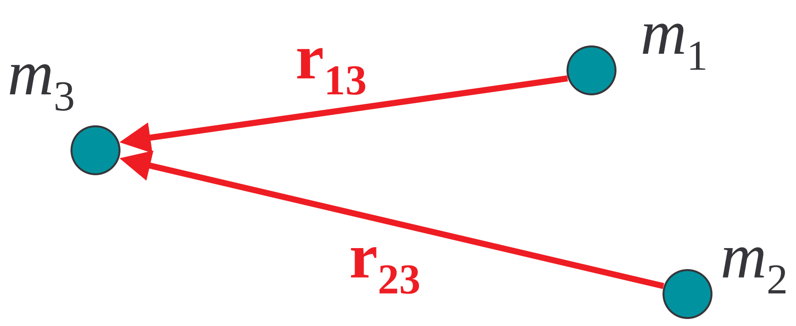
Lecture 37: Superposition and the Shell Theorems

- Last time, we saw that Newton could explain both the fall of an apple near the Earth and the motion of the moon around the Earth if we assumed that a universal force called "gravity" acted between all pairs of objects with the form:

$$\mathbf{F}_{12} = \frac{-Gm_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

- But what if there are more than two objects in the system?
  - What is the force on m_3 below?

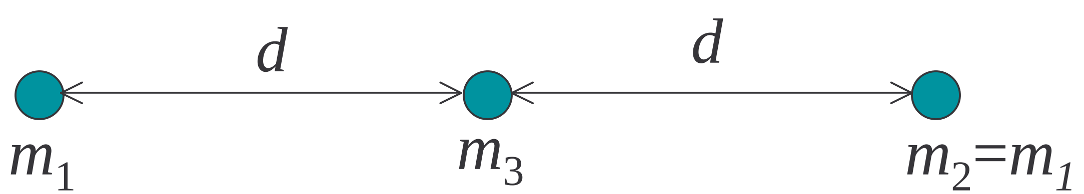


- We can simply add up the forces of all the two-body interactions involving m_3. This is called *superposition*:

$$\mathbf{F_3} = -\frac{Gm_1m_3}{r_{13}}\hat{\mathbf{r}}_{\mathbf{13}} - \frac{Gm_1m_2}{r_{12}}\hat{\mathbf{r}}_{12}$$

Note that this is a vector equation!

- For example, the force on m_3 in the situation below would be 0:

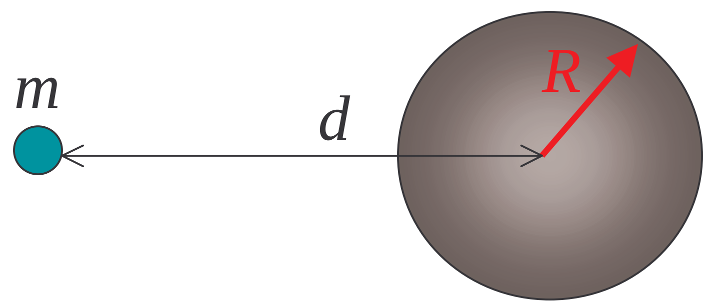


- Note that this is *not* the force that the $m_1 + m_2$ system would exert if we assumed all the mass was at the center of mass of that system
  - That force would be infinite!

- Superposition seems like a natural, elegant way to deal with the gravitational attraction of multiple objects
- But it was a huge problem for Newton
  - In his analysis of the moon and the apple, in which the ideas of gravity were developed, he had assumed that the Earth acted as a point particle with all its mass at the center
  - The Earth is far enough from the moon that this might not be a bad approximation
  - However, it's clearly a bad approximation for the apple! And Newton knew that in the general case a system of particles doesn't exert gravity as though all the mass were at the center of mass
- So he needed to determine how big a correction to make to his equations to account for the fact that the Earth is not a point particle

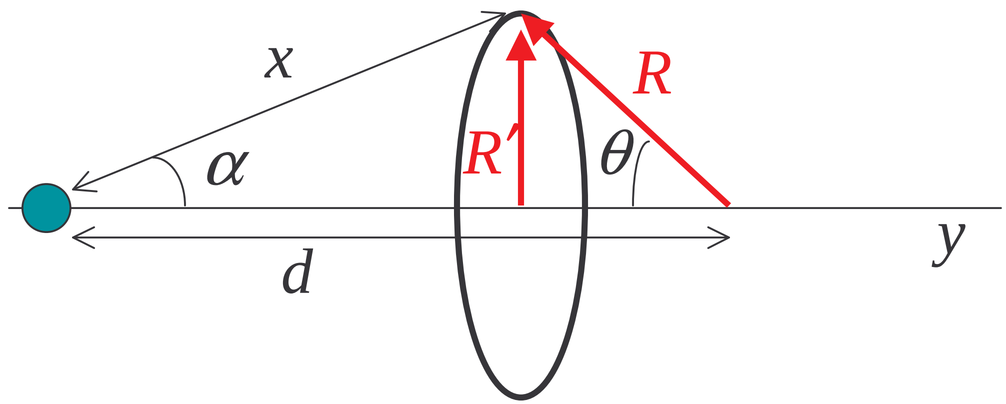
The Shell Theorems

- So he wanted to find the exact gravitational force on *m* for the situation below:



- In other words, he needed to apply superposition to every little piece of the spherical mass
  - This is the problem for which he invented calculus

- To begin, Newton realized that the sphere could be broken up into thin shells nested inside of each other. Further, each shell could be broken up into a collection of thin rings. So the first problem to solve was this:



- What is the force on *m* from the ring?
- Symmetry tells us that the net force must be along the *y* direction
- The *y* component of force from a small element of mass d*m* is:

$$dF = \frac{Gm\,dm}{x^2}\cos\alpha$$

- Therefore the total force from the ring is:

$$F_{\text{ring}} = \int \mathrm{d}F = \int \frac{Gm\mathrm{d}m}{x^2}\cos\alpha = \frac{Gm}{x^2}\cos\alpha \int \mathrm{d}m$$

$$= \frac{Gm}{x^2}\cos\alpha m_{\text{ring}}$$

- Now we assume our ring has a thickness *t* and spans an angle of dθ as viewed from the center of the sphere
- This means that the mass of the ring is:

$$m_{\text{ring}} = \rho V_{\text{ring}} = \rho 2\pi R' t R \mathrm{d}\theta$$

$$= \rho 2\pi R \sin\theta t R \mathrm{d}\theta = 2\pi t \rho R^2 \sin\theta \mathrm{d}\theta$$

where ρ is the density of the shell

- Now we are poised to find the total force from the shell, by taking m_{ring} to be dm and adding up all the contributions:

$$F_{\text{shell}} = \int \mathrm{d}F_{\text{ring}} = \int \frac{Gm}{x^2} \cos\alpha \,\mathrm{d}m$$

$$= \int \frac{Gm}{x^2} \cos\alpha \left(2\pi t \rho R^2 \sin\theta \,\mathrm{d}\theta\right)$$

$$= 2\pi t \rho R^2 Gm \int \frac{\cos\alpha \sin\theta \,\mathrm{d}\theta}{x^2}$$

- The variables α and θ depend on x. We need to quantify this dependence to complete the integral
- First off, we see that:

$$\cos\alpha = \frac{d - R\cos\theta}{x}$$

- Using the Law of Cosines, we can show that:

$$R\cos\theta = \frac{d^2 + R^2 - x^2}{2d}$$

And we can differentiate this to find:

$$-R\sin\theta \mathrm{d}\theta = \frac{-2x\mathrm{d}x}{2d} = \frac{-x\mathrm{d}x}{d}$$

$$\sin\theta \mathrm{d}\theta = \frac{x}{dR}\mathrm{d}x$$

- Now we can express the entire integral in terms of *x*:

$$\int \frac{\cos\alpha \sin\theta \mathrm{d}\theta}{x^2} = \int \frac{\left(\dfrac{d - \dfrac{d^2 + R^2 - x^2}{2d}}{x}\right)\left(\dfrac{x}{dR}\mathrm{d}x\right)}{x^2}$$

$$
\begin{aligned}
F_{\text{shell}} &= 2\pi t\rho R^2 Gm \int_{d-R}^{d+R} \frac{d^2 - R^2 + x^2}{2d^2 R x^2}\,\mathrm{d}x \\
&= \frac{\pi t\rho R^2 Gm}{d^2 R} \int_{d-R}^{d+R} \left(\frac{d^2 - R^2}{x^2} + 1 \right) \mathrm{d}x \\
&= \frac{\pi t\rho RGm}{d^2} \left[\frac{R^2 - d^2}{x} + x \right]_{d-R}^{d+R} \\
&= \frac{\pi t\rho RGm}{d^2} \left[R - d + R + d + d + R - d + R \right] \\
&= \frac{\pi t\rho RGm}{d^2} \cdot 4R
\end{aligned}
$$

- Now we note that the total mass of the shell is given by:

$$
m_{\text{shell}} = 4\pi R^2 t\rho
$$

- This means that:

$$F_{\text{shell}} = \frac{Gmm_{\text{shell}}}{d^2}$$

- The shell really does act as though all the mass is concentrated at the center!
- And since the spherical Earth is nothing more than a collection of shells, Newton's assumption that even the apple can be treated as though only a point mass at the center of the Earth is acting on it is exactly true!
- Note that this result holds for a sphere even if the density varies as a function of radius

The Second Shell Theorem

- What about the force felt by a mass *inside* a uniform shell?
- All that changes in our derivation is the lower limit of integration:

$$F_{\text{shell}} = \frac{\pi t \rho R G m}{d^2}\left[\frac{R^2 - d^2}{x} + x\right]_{R-d}^{d+R}$$

$$= \frac{\pi t \rho R G m}{d^2}\left[R - d - R - d + d + R - R + d\right]$$

$$= 0$$

- So inside a shell one feels no gravity at all!