Lecture 37: Motion of a Symmetric Top

• Last time, we began exploring the motion of a symmetric top when no external torque is applied
• We found that $\omega_3$ (the component along the axis of symmetry) was constant
• The equations of motion for $\omega_1$ and $\omega_2$ can be expressed as:

$$\dot{\eta} - i\Omega \eta = 0$$

where $\eta = \omega_1 + i\omega_2$
• That’s an equation we know how to solve:

$$\eta(t) = Ae^{i(\Omega t + \delta)}$$

• To make life easy, we’ll choose the phase $\delta$ to be zero
• Writing the equation in terms of the components of $\omega$ gives:

$$\omega_1 + i\omega_2 = A e^{i\Omega t}$$

• Writing the real and imaginary parts of the above equation separately gives:

$$\omega_1 = A \cos(\Omega t)$$
$$\omega_2 = A \sin(\Omega t)$$

• To interpret this result, note that the magnitude of the angular velocity vector must be constant:

$$|\omega| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 \cos^2(\Omega t) + A^2 \sin^2(\Omega t) + \omega_3^2}$$
$$= \sqrt{A^2 + \omega_3^2} = \text{const}$$
• In other words, in the body frame the motion would look like:

• How does the motion look in an inertial frame?
  – We start by noting that both \( \mathbf{L} \) and rotational kinetic energy are constant

• So, \( \mathbf{L} \) is pointed along a fixed direction in the inertial frame, which we can take to be the \( x^3 \) direction

This conical motion of the angular velocity is called “precession”
• The kinetic energy can be expressed as $T = \frac{1}{2} \mathbf{\omega} \cdot \mathbf{L}$, which implies that the component of $\mathbf{\omega}$ along the $\mathbf{L}$ direction must be constant.

• In other words, the motion must look like:

• Further, we can show that $\mathbf{L}$, $\mathbf{\omega}$, and the $x_3$ axis all must lie in the same plane:
  - First, note that: $\mathbf{\omega} \times \mathbf{e}_3 = \omega_2 \mathbf{e}_1 - \omega_1 \mathbf{e}_2 = \mathbf{A}$
  - $\mathbf{A}$ is a vector perpendicular to the plane of $\mathbf{\omega}$ and $\mathbf{e}_3$

$$\mathbf{L} \cdot \mathbf{A} = L_1 A_1 + L_2 A_2 = I_1 \omega_1 \omega_2 - I_2 \omega_2 \omega_1 = 0 \text{ (since } I_1 = I_2)$$
• Combining all of this information, we see that the geometry must be:

• The body cone can be thought of as rolling without slipping on the surface of the space cone
Motion of a Top Under Gravity, with One Fixed Point

• We now consider the motion observed with a child’s toy top
• Again we assume there is an axis of symmetry, such that
  \[ I_1 = I_2 \]
• Take the fixed point to be the origin of both the inertial and body reference frames
• Assume the center of mass of the top is a distance \( h \) from the fixed point
  – So the potential energy is
    \[ U = Mgh \cos \theta \]
  – The kinetic energy about the fixed point is purely rotational:
    \[ T = \frac{1}{2} \sum_i I_i \omega_i^2 \]
• So the Lagrangian is simply:

\[ L = \frac{1}{2} \sum_i I_i \omega_i^2 - Mgh \cos \theta \]

• But we need to express this in terms of our generalized coordinates – the Eulerian angles
  – Luckily, we already know how to write the \( \omega_i \) in terms of \( \theta, \phi, \) and \( \psi \) (and their time derivatives)

• After a little algebra (see p. 457 of the text) we find:

\[ L = \frac{1}{2} I_1 \left( \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right)^2 - Mgh \cos \theta \]

• The first thing to note is that neither \( \phi \) nor \( \psi \) appear in the Lagrangian
  – So the momenta conjugate to these quantities must be conserved
\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\phi} + I_3 \dot{\psi} \cos \theta = \text{const} \]

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos \theta) = I_3 \omega_3 = \text{const} \]

- Solving the above for \( \dot{\phi} \) gives:

\[ \dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta} \]

- Precession again (this time about the vertical direction)
Motion in $\theta$

- The motion in $\theta$ is the most counterintuitive feature of a top’s motion
  - i.e., the top doesn’t fall over!

- We can see why simply by considering conserved quantities. First, energy is conserved (in the real world there’s friction, so tops eventually slow down, but we’re ignoring that):

\[
E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 + Mgh \cos \theta
\]

\[
= \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + Mgh \cos \theta = \text{const}
\]

- But,

\[
I_3 \omega_3^2 = I_3 \left( \frac{p_\psi}{I_3} \right)^2 = \frac{p_\psi^2}{I_3} = \text{const.}
\]
• Thus, the quantity:

\[ E' = \frac{1}{2} I_1 \left( \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + Mgh \cos \theta \]

\[ = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + Mgh \cos \theta \]

is also constant

• We can write this as:

\[ E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta) \]

• This looks like the expression for energy in one dimension, with \( V(\theta) \) playing the role of an “effective potential”
• A “typical” $V(\theta)$ looks like:

![Graph showing $V(\theta)$ with an allowed region at $E'$]

- For a given $E'$ there is an allowed region in $\theta$
- The body will oscillate within the allowed range
  - This oscillation is called *nutation*