Symmetric Top: Motion in $\theta$

- The motion in $\theta$ is the most counterintuitive feature of a top’s motion
  - i.e., the top doesn’t fall over!
- We can see why simply by considering conserved quantities. First, energy is conserved (in the real world there’s friction, so tops eventually slow down, but we’re ignoring that):

\[
E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 + Mgh \cos \theta
\]

\[
= \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + Mgh \cos \theta = \text{const}
\]

- But,

\[
I_3 \omega_3^2 = I_3 \left( \frac{p_\psi}{I_3} \right)^2 = \frac{p_\psi^2}{I_3} = \text{const.}
\]
Thus, the quantity:

\[ E' = \frac{1}{2} I_1 \left( \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + Mgh \cos \theta \]

\[ = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + Mgh \cos \theta \]

is also constant.

We can write this as:

\[ E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta) \]

This looks like the expression for energy in one dimension, with \( V(\theta) \) playing the role of an “effective potential”
• A “typical” $V(\theta)$ looks like:

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    axis lines = center,
    xlabel = $\theta$,
    ylabel = $V$,
    xmin = -0.36, xmax = 3.14,
    ymin = 0, ymax = 20,
    no markers,
    domain = -0.36:3.14,
]
\addplot[variable = \theta, smooth, thick, black] {20*(1 - cos(\theta))^2};
\addplot[variable = \theta, smooth, thick, red] {14};
\end{axis}
\end{tikzpicture}
\end{center}

• For a given $E'$ there is an allowed region in $\theta$
• The body will oscillate within the allowed range
  – This oscillation is called *nutation*
Stability of Rigid-Body Rotations

• We start with any rigid body
  – i.e., no longer assuming that two of the principal moments are the same. Let’s label the principal moment in order of their size: $I_1 < I_2 < I_3$
• Assume that in the initial state the body is rotating about the principal axes with smallest moment
  \[ \mathbf{\omega} = \omega_1 \mathbf{e}_1 \]
• We want to explore what happens when the body is given a slight nudge
  – Aside from this nudge, we’ll assume no external forces act on the body
• Immediately after the nudge, the angular velocity vector is:
  \[ \mathbf{\omega} = \omega_1 \mathbf{e}_1 + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 \quad \lambda \text{ and } \mu \text{ are small} \]
• Applying Euler’s Equations for force-free motion to this situation, we find:

\[
\begin{align*}
(I_2 - I_3) \lambda \mu - I_1 \dot{\omega}_1 &= 0 \\
(I_3 - I_1) \mu \omega_1 - I_2 \dot{\lambda} &= 0 \\
(I_1 - I_2) \lambda \omega_1 - I_3 \dot{\mu} &= 0
\end{align*}
\]

• The product \( \mu \lambda \) in the first equation is negligibly small, so \( \omega_1 \) is constant

• Now consider how \( \mu \) and \( \lambda \) change with time
  – If they remain small, the motion will look pretty much like the initial state – rotation about the \( x_1 \) axis
  – If they tend to large values, the motion will come look significantly different from the initial motion

• In other words, the motion is either stable or unstable
• Rearranging the second two Euler Equations gives:

\[
\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2} \omega_1\right) \mu
\]

\[
\dot{\mu} = \left(\frac{I_1 - I_2}{I_3} \omega_1\right) \lambda
\]

• To solve this system, take the time derivative of the first equation:

\[
\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2} \omega_1\right) \dot{\mu}
\]

• Now substitute the expression for \( \dot{\mu} \) from above:

\[
\ddot{\lambda} = -\left(\frac{I_1 - I_3}{I_2} \omega_1\right) \left(\frac{I_1 - I_2}{I_3} \omega_1\right) \lambda
\]
• Everything on the right-hand side in front of the $\lambda$ is a constant (which we define as $-\Omega^2$), so we have:

$$\ddot{\lambda} = -\Omega^2 \lambda; \quad \Omega = \omega_1 \sqrt{\left(\frac{I_1 - I_3}{I_2}\right)\left(\frac{I_1 - I_2}{I_3}\right)}$$

• The general solution to this equation is:

$$\lambda(t) = Ae^{i\Omega t} + Be^{-i\Omega t}$$

– we’d find a similar result for $\mu$

• This means that the time evolution of $\lambda$ depends on whether $\Omega$ is imaginary or real:

1. If $\Omega$ is real: solution describes oscillatory motion – $\lambda$ never becomes large. Rotation is stable.

2. If $\Omega$ is imaginary: the second term represents exponential growth in $\lambda$. Rotation is unstable
• Which case applies here? We assumed from the start that the $x_1$ axis had the smallest principal moment 
• Therefore, both $I_1 - I_3$ and $I_1 - I_2$ are negative
  – The product is positive, and therefore $\Omega$ is real
  – So the rotation is stable!
• If the $x_1$ axis had the largest principal moment, both $I_1 - I_3$ and $I_1 - I_2$ would be positive
  – Again, stable rotation
• On the other hand, if the $x_1$ axis has the intermediate moment of inertia, the product $(I_1 - I_3)(I_1 - I_2)$ must be negative
  – In this case the rotation is unstable
What if two of the principal moments are equal?

• Let’s say $I_1 = I_2$. Then the Euler equations would be:

$$\dot{\lambda} = \left( \frac{I_3 - I_1}{I_2} \omega_1 \right) \mu$$

$$\dot{\mu} = \left( \frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda = 0$$

so $\mu$ is a constant. That means we can directly integrate the differential equation for $\lambda$ to find:

$$\lambda(t) = C + Dt$$

• So $|\lambda|$ becomes large, meaning rotation about $x_1$ is unstable
  – Same result is obtained for rotation about $x_2$

• Only rotations about the non-equal (symmetry) axis are stable