Lecture 39: Coupled Oscillations

• By now you are experts on the motion that occurs when a mass is connected to an ideal spring  
  – i.e., simple harmonic motion
• But what about the case where there are more masses and more springs?
• Let’s start with the example illustrated below:

\[
\begin{align*}
  m_1 &= M \\
  m_2 &= M \\
  k_{12} &= 2k
\end{align*}
\]

Start with all three springs at their equilibrium lengths  
Then move \( m_1 \) a distance \( x_1 \) and move \( m_2 \) a distance \( x_2 \)
• Then both masses will feel forces due to the two springs connected to it. Using Hooke’s Law we find the magnitude of these forces:

On $m_1$:  
$$F_1 = M \ddot{x}_1 = -kx_1 - k_{12} (x_1 - x_2)$$

On $m_2$:  
$$F_2 = M \ddot{x}_2 = -kx_2 - k_{12} (x_2 - x_1)$$

• This means we have the following equations of motion for the system:

$$M \ddot{x}_1 + (k + k_{12}) x_1 - k_{12} x_2 = 0$$

$$M \ddot{x}_2 + (k + k_{12}) x_2 - k_{12} x_1 = 0$$

• We need to find the functions $x_1(t)$ and $x_2(t)$ that satisfy both of the above equations
• Since our experience is that masses attached to springs oscillate, let’s try a solution like:

\[ x_1(t) = B_1 e^{i\omega t} \]
\[ x_2(t) = B_2 e^{i\omega t} \]

Note that \( B_1 \) and \( B_2 \) are complex numbers – the real parts of \( x_1 \) and \( x_2 \) will describe the motion

• Plugging these guesses back into the differential equation gives:

\[-M \omega^2 B_1 e^{i\omega t} + (k + k_{12}) B_1 e^{i\omega t} - k_{12} B_2 e^{i\omega t} = 0\]
\[-M \omega^2 B_2 e^{i\omega t} + (k + k_{12}) B_2 e^{i\omega t} - k_{12} B_1 e^{i\omega t} = 0\]

which reduces to:

\[ (k + k_{12} - M \omega^2) B_1 - k_{12} B_2 = 0 \]
\[-k_{12} B_1 + (k + k_{12} - M \omega^2) B_2 = 0 \]
• This system of equations has a solution only when:

\[
\begin{vmatrix}
  k + k_{12} - M \omega^2 & -k_{12} \\
  -k_{12} & k + k_{12} - M \omega^2
\end{vmatrix}
= 0
\]

\[
(k + k_{12} - M \omega^2)^2 - k_{12}^2 = 0
\]

\[
\omega = \pm \sqrt{\frac{k + k_{12} \pm k_{12}}{M}}
\]

• Just as we found principal moments of inertia for a rigid system, we now have found characteristic (or “eigen”) frequencies for the coupled oscillators:

\[
\omega_1 = \sqrt{\frac{k + 2k_{12}}{M}}; \quad \omega_2 = \sqrt{\frac{k}{M}}
\]
• So the general solution looks like:

\[ x_1(t) = B_{11}^+ e^{i\omega_1 t} + B_{11}^- e^{-i\omega_1 t} + B_{12}^+ e^{i\omega_2 t} + B_{12}^- e^{-i\omega_2 t} \]

\[ x_2(t) = B_{21}^+ e^{i\omega_1 t} + B_{21}^- e^{-i\omega_1 t} + B_{22}^+ e^{i\omega_2 t} + B_{22}^- e^{-i\omega_2 t} \]

• That’s a lot of coefficients! But let’s assume all the coefficients in front of the \( e^{-i\omega_1 t} \) and \( e^{\pm i\omega_2 t} \) terms are zero. Then we have:

\[ x_1(t) = B_{11}^+ e^{i\omega_1 t} + B_{11}^- e^{-i\omega_1 t} \]

\[ x_2(t) = B_{21}^+ e^{i\omega_1 t} + B_{21}^- e^{-i\omega_1 t} \]

• Plugging these back into our system of differential equations gives:

\[ (k + k_{12} - M \omega_1^2) B_{11}^+ e^{i\omega t} - k_{12} B_{21}^+ e^{i\omega t} = 0 \]

\[ (k + k_{12} - M \omega_1^2) B_{21}^+ e^{i\omega t} - k_{12} B_{11}^+ e^{i\omega t} = 0 \]
• Using the known value of $\omega_1$, this becomes:

\[
\begin{align*}
&\left( k + k_{12} - M \left[ \frac{k + 2k_{12}}{M} \right] \right) B_{11}^+ - k_{12} B_{21}^+ = 0 \\
&\left( k + k_{12} - M \left[ \frac{k + 2k_{12}}{M} \right] \right) B_{21}^+ - k_{12} B_{11}^+ = 0 \\
&-k_{12} B_{11}^+ - k_{12} B_{21}^+ = 0 \\
&B_{11}^+ = -B_{21}^+
\end{align*}
\]

• Similarly, we can show that $B_{11}^- = -B_{21}^-$, $B_{12}^+ = B_{22}^+$, and $B_{12}^- = B_{22}^-$

• So the general solution reduces to

\[
\begin{align*}
x_1(t) &= B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \\
x_2(t) &= B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t}
\end{align*}
\]
• Let’s do the same problem again, with a different set of coordinates:

\[ \eta_1 = x_1 - x_2 \]
\[ \eta_2 = x_1 + x_2 \]

which means

\[ x_1 = \frac{1}{2}(\eta_1 + \eta_2) \text{ and } x_2 = \frac{1}{2}(\eta_2 - \eta_1) \]

• In terms of the \( \eta \)'s, the equations of motion are

\[ M (\ddot{\eta}_1 + \ddot{\eta}_2) + (k + k_{12})(\eta_1 + \eta_2) - k_{12}(\eta_1 - \eta_2) = 0 \]
\[ M (\ddot{\eta}_1 - \ddot{\eta}_2) + (k + k_{12})(\eta_1 - \eta_2) - k_{12}(\eta_1 + \eta_2) = 0 \]
• Adding these equations gives:
  \[ M \ddot{\eta}_1 + (k + k_{12}) \eta_1 - k_{12} \eta_1 = 0 \]
  \[ M \ddot{\eta}_1 + k \eta_1 = 0 \]
  and subtracting them gives:
  \[ M \ddot{\eta}_2 + (k + k_{12}) \eta_2 + k_{12} \eta_2 = 0 \]
  \[ M \ddot{\eta}_2 + (k + 2k_{12}) \eta_2 = 0 \]

• So we see that the motion is just simple harmonic in these coordinates
  – And the oscillation frequencies are the characteristic frequencies we found earlier
• $\eta_1$ and $\eta_2$ are called “normal coordinates” for this system

• What do they mean physically?
  – Choose a special set of initial conditions:
    \begin{align*}
    x_1(0) &= -x_2(0) \\
    \dot{x}_1(0) &= -\dot{x}_2(0)
    \end{align*}
  – These imply that
    \begin{align*}
    \eta_2(0) &= \dot{\eta}_2(0) = 0
    \end{align*}
  – Just like any simple harmonic oscillator that begins at rest at its equilibrium point, the $\eta_2$ “oscillator” will remain at rest for all time

• In other words, the motion for this set of initial conditions is represented by oscillation in $\eta_1$
• What does oscillation in $\eta_1$ look like?
  – It means the *distance* between the two masses increases and decreases harmonically, but the average position remains fixed at 0
  – Looks like antisymmetrical oscillation
• We could also start with the following initial conditions:

\[ x_1(0) = x_2(0) \]
\[ \dot{x}_1(0) = \dot{x}_2(0) \]

• Then the motion is simple harmonic oscillation in $\eta_2$
  – The average position changes harmonically, but the distance between the masses is fixed
  – Looks like symmetrical oscillation
• Note that:
  – this example contains many of the features shared by any system of coupled oscillators
    • we can always find eigenfrequencies and normal coordinates
  – the eigenfrequency associated with the antisymmetric mode is larger than that associated with the symmetric mode
    • this is true for all systems of coupled oscillators
    • for systems with more masses/springs, there will be more eigenmodes, each with a different degree of symmetry – but the most symmetric one will have the lowest eigenfrequency
  – the general solution (for any set of initial conditions) is a linear combination of the eigenmodes