2 Vectors, Tensors, Scalars

Having defined different basis vectors, we can express the components of any vector $\vec{A}$ with respect to either of them. When we use the coordinate basis vectors, i.e.,

$$\vec{A} = A^i \hat{e}_i$$  \hspace{1cm} (21)

we use a superscript (or upstairs) notation for the components, which we call the \textit{contravariant} components of the vector. On the other hand, when we use the dual basis vector, i.e.,

$$\vec{A} = A_i \hat{e}^i$$  \hspace{1cm} (22)

we use a subscript (or downstairs) notation for the components, which we call the \textit{covariant} components of the vector $\vec{A}$.

The contravariant and covariant components of a vector transform in different ways from one coordinate system to another. In order to study their transformations, we will consider two coordinate systems $\xi^i$ and $\xi'^j$ and use unprimed and primed quantities, respectively, to denote the various vector components in the two systems.

We will start by inserting the definition of the coordinate basis (12) into equation (21), i.e.,

$$\vec{A} = A^i \hat{e}_i = A^i \frac{\partial \vec{r}}{\partial \xi^i}$$  \hspace{1cm} (23)

We will then perform a change of coordinates in the derivatives using the chain rule,\footnote{Remember that $\xi'^j = \xi'^j(\xi)$ and $\xi^i = \xi^i(\xi')$.}

$$\vec{A} = A^i \frac{\partial \vec{r}}{\partial \xi^i} \frac{\partial \xi'^j}{\partial \xi^i} = A^i \frac{\partial \xi'^j}{\partial \xi^i} \hat{e}^j.$$  \hspace{1cm} (24)

In the last equation we have used again the definition (12) but for the primed coordinate frame. Comparing the last term with the definition (21) we finally find

$$A'^j = A^i \frac{\partial \xi'^j}{\partial \xi^i}.$$  \hspace{1cm} (25)

Transformation of Contravariant Components

In order to derive the transformation rule for the covariant components of a vector, we will use an auxiliary Cartesian coordinate system $x^i$ with basis vectors $\hat{e}^*^i$. We will start again by inserting the definition of the dual basis (13) into equation (22) and use the chain rule to perform a change of coordinates in the derivatives. In detail

$$\vec{A} = A_i \hat{e}^i = A_i \frac{\partial x^j}{\partial \xi^i} \hat{e}^*^j = A_i \frac{\partial \xi'^k}{\partial x^j} \frac{\partial x^j}{\partial \xi^i} \hat{e}^*^k = A_i \frac{\partial \xi'^k}{\partial \xi^i} e'^k.$$  \hspace{1cm} (26)

Comparing the last term with the definition (22) and simply changing the dummy index for $k$ to $j$, we obtain

$$A'_j = A_i \frac{\partial \xi'^j}{\partial \xi^i}.$$  \hspace{1cm} (27)

Transformation of Covariant Components

Note in these transformations how useful the notation of subscripts and superscripts has been.
Until this point, we have assumed that the space on which we have defined coordinates, basis vectors, and vector components is flat. Extending these definitions to a general curved space will require a different understanding of the various quantities involved. For example, in a curved space, directed line segments can only be infinitesimally short and, therefore, the position vector \( \vec{r} \) that we used in the definition of the coordinate basis is not well defined. Even though there is a way of extending all these geometric interpretations in curved spaces, it is sufficient for the purposes of this class to take a somewhat backward approach.

We will define as the contravariant components of a vector in an \( N \)-dimensional space, and denote them with superscript notation, an ordered set of \( N \) physical quantities (i.e., components of velocities, momenta, fields) that transform between coordinate systems according to equation (25).

Similarly, we will define as the covariant components of a vector in an \( N \)-dimensional space, and denote them with subscript notation, an ordered set of \( N \) physical quantities (i.e., components of velocities, momenta, fields) that transform between coordinate systems according to equation (27).

In general, we will define as a tensor and denote by

\[
T^{ijkl... \alpha \beta \gamma \delta...} 
\]

an ordered set of physical quantities, some of which transform according to the rules for contravariant components (and we will use superscript notation) and some of which transform according to the rule for covariant components, i.e.,

\[
T^{ijkl... \alpha \beta \gamma...} = \frac{\partial \xi'^i}{\partial \xi^j} \frac{\partial \xi'^j}{\partial \xi^k} \frac{\partial \xi'^k}{\partial \xi^l} \frac{\partial \xi'^l}{\partial \xi^m} ... T^{ijkl... \alpha \beta \gamma...} .
\]

The total number of indices is called the rank of the tensor. A vector is a tensor of rank one. A scalar is a tensor of rank zero.

**Example:** The inner product of two vectors

In this example, we will derive some useful expressions for the inner product between two vectors and show that it is a scalar quantity, i.e., that it is invariant between coordinate transformations.

We will start with two vectors,

\[
\vec{A} = A^i \hat{e}_i \\
\vec{B} = B^j \hat{e}_j
\]

and calculate their product as

\[
\vec{A} \cdot \vec{B} = (A^i \hat{e}_i) \cdot (B^j \hat{e}_j) = A^i B_j \delta^j_i = A^i B_i .
\]

Here we used the fact that \( \hat{e}_i \cdot \hat{e}_j = \delta^j_i \). We can follow the exact same procedure using the covariant components of vector \( \vec{A} \) and the contravariant components of vector \( \vec{B} \). The final set of expressions for the inner product of two vectors is

\[
\vec{A} \cdot \vec{B} = A^i B_i = A_i B^i
\]
This last expression proves that the inner product of two vectors is a scalar quantity.

**Example:** Projecting onto basis vectors

In an orthonormal coordinate system, e.g., in a Cartesian system \((\hat{e}_x, \hat{e}_y, \hat{e}_z)\), we can calculate the component of a vector \(\vec{A}\) along one of the coordinate lines using inner products of the form \(A_x = \vec{A} \cdot \hat{e}_x\). This is not, of course, the case if the system is non-orthogonal. However, the definitions of the coordinate and dual basis provide us with a very useful tool in calculating contravariant and covariant components of vectors, independent of whether the coordinate system is orthogonal or not.

Starting with the definition of the contravariant components of a vector \(\vec{A}\) and multiplying both sides of the equation with a dual basis vector \(\hat{e}_i\) we obtain

\[
\vec{A}_i = \vec{A} \cdot \hat{e}_i \Rightarrow A^i (\hat{e}_j \cdot \hat{e}_i) = A^j \delta^i_j
\]

and, therefore,

\[
A^i = \vec{A} \cdot \hat{e}_i
\]

Similarly, we can also prove that

\[
A_i = \vec{A} \cdot \hat{e}_i
\]

### 3 The Metric Tensor

We will use the dot product of two vectors, and in particular of two basis vectors, in order to specify the geometry of a general curved space. Starting from

\[
\vec{A} \cdot \vec{B} = (A^i \hat{e}_i) \cdot (B^j \hat{e}_j) = A^i B^j (\hat{e}_i \cdot \hat{e}_j)
\]

we only need to specify the elements of the rank-2 covariant tensor

\[
g_{ij} = \hat{e}_i \cdot \hat{e}_j
\]

which we will call the metric tensor.

If, instead of the product of two vectors, we calculate the product of an infinitesimal translational vector to itself, i.e.,

\[
ds^2 = d\vec{x} \cdot d\vec{x} = g_{ij} dx^i dx^j
\]

we call the result the line element of the space.

We can also define the contravariant components of the metric tensor as

\[
g^{ij} = \hat{e}^i \cdot \hat{e}^j
\]

and the mixed components as

\[
g^{i j} = \hat{e}^i \cdot \hat{e}_j
\]

Because of the orthogonality of the coordinate and dual basis vectors, \(g^{ij} = \delta^i_j\).

The metric tensor has very many uses in problem solving, one of which is to help us transform the components of a tensor between the coordinate and dual basis (i.e.,
to raise on lower indices). For example, we show earlier that the inner product of two vectors is equal to
\[ \vec{A} \cdot \vec{B} = A^i B_i \, . \] (43)

However, we can write the same inner product using the definition of the metric tensor as
\[ \vec{A} \cdot \vec{B} = A^i B^j g_{ij} \, . \] (44)

Comparing these two equations we obtain
\[ B_i = g_{ij} B^j \, . \] (45)

Similarly, we can prove that
\[ B^i = g^{ij} B_j \, . \] (46)

Finally, we can use this last property of the metric tensor to prove that \( g_{ij} \) is the inverse of \( g^{ij} \). We will start from the dot product of a coordinate to a dual basis vector,
\[ \hat{e}^i \cdot \hat{e}_k = \delta^i_k \Rightarrow (g^{ij} \hat{e}_j) \cdot \hat{e}_k = \delta^i_k \Rightarrow g^{ij} (\hat{e}_j \cdot \hat{e}_k) = \delta^i_k \] (47)

from which we obtain
\[ g^{ij} g_{jk} = \delta^i_k \, . \] (48)
**Useful Expressions**

Coordinate Basis Vectors
\[ \hat{e}_i \equiv \frac{\partial \vec{r}}{\partial \xi^i} \]  
(49)

Dual Basis Vectors
\[ \check{e}^i \equiv \vec{\nabla} \xi^i \]  
(50)

Orthogonality of Basis Vectors
\[ \hat{e}_i \check{e}^j = \delta_i^j \]  
(51)

Contravariant Components of Vector
\[ \vec{A} = A^i \hat{e}_i \]  
(52)

Covariant Components of Vector
\[ \vec{A} = A_i \check{e}^i \]  
(53)

Transformation of Contravariant Components
\[ A'^i = A^i \frac{\partial \xi' j}{\partial \xi_i} \]  
(54)

Transformation of Covariant Components
\[ A'_j = A_i \frac{\partial \xi^i}{\partial \xi' j} \]  
(55)

Transformation of General Tensor Components
\[ T'^{i j k \ldots}_{\alpha \beta \gamma \ldots} = \frac{\partial \xi'^{i}}{\partial \xi^i} \frac{\partial \xi'^{j}}{\partial \xi^j} \ldots \frac{\partial \xi'^{A}}{\partial \xi^A} \frac{\partial \xi'^{B}}{\partial \xi^B} \ldots T^{i j k \ldots}_{A B \ldots} \]  
(56)

Inner Product of Vectors
\[ \vec{A} \cdot \vec{B} = A^i B_i = A_i B^i \]  
(57)

Vector Components
\[ A^i = \vec{A} \cdot \hat{e}^i \]  
\[ A_i = \vec{A} \cdot \check{e}_i \]  
(58)

The Metric Tensor
\[ g_{ij} = \hat{e}_i \cdot \hat{e}_j \]  
(60)

Other components of the metric tensor
\[ g_{ij} g^{jk} = \delta_i^k \]  
\[ g^{i j} = \delta^i_j \]  
(61)

Lowering and Raising an Index
\[ B_i = g_{ij} B^j \quad B^i = g^{ij} B_j \]  
(63)