Lecture #28

The deformation tensor II

Even though the deformation tensor is defined using Lagrangian coordinates, it is often simpler to evaluate it using the corresponding Eulerian coordinates.

Let a point \( M \) with coordinates \( x(x_1, x_2, x_3) \) before deformation, which moves to a new Eulerian position \( \vec{r}'(x_1', x_2', x_3') \) after the deformation. Let

\[
\vec{r}' = \vec{r} + \vec{w}
\]

where \( \vec{w} \) is the displacement vector.

For a pure translation of the body, \( \vec{w} \) is the same for all points. However, for a deformation, \( \vec{w} \) depends on \( (x_1, x_2, x_3) \).

In order to calculate the deformation tensor, we need to discuss the basis vectors in the Lagrangian frame.
By definition
\[ \hat{e}_i = \frac{\partial \hat{r}}{\partial \hat{x}_i} \]
and
\[ \hat{e}_i' = \frac{\partial \hat{r}'}{\partial \hat{x}_i} \]
where \( \hat{e}_i \) and \( \hat{e}_i' \) are the basis vectors in the
Lagrangian frame before and after deformation.
But
\[ \hat{r}' = \hat{r} + \hat{\omega} \Rightarrow \]
\[ \frac{\partial \hat{r}'}{\partial \hat{x}_i} = \frac{\partial \hat{r}}{\partial \hat{x}_i} + \frac{\partial \hat{\omega}}{\partial \hat{x}_i} \Rightarrow \hat{e}_i' = \hat{e}_i + \frac{\partial \hat{\omega}}{\partial \hat{x}_i} \]
and hence
\[ \hat{e}_i' \cdot \hat{e}_j' = \left( \hat{e}_i + \frac{\partial \hat{\omega}}{\partial \hat{x}_i} \right) \left( \hat{e}_j + \frac{\partial \hat{\omega}}{\partial \hat{x}_j} \right) = \]
\[ = \hat{e}_i \cdot \hat{e}_j + \hat{e}_i \cdot \frac{\partial \hat{\omega}}{\partial \hat{x}_j} + \hat{e}_j \cdot \frac{\partial \hat{\omega}}{\partial \hat{x}_i} + \frac{\partial \hat{\omega}}{\partial \hat{x}_i} \cdot \frac{\partial \hat{\omega}}{\partial \hat{x}_j} \]
As a result, the deformation tensor becomes
\[ \varepsilon_{ij} = \frac{1}{2} \left[ \hat{e}_i \frac{\partial \hat{\omega}}{\partial \hat{x}_j} + \hat{e}_j \frac{\partial \hat{\omega}}{\partial \hat{x}_i} + \frac{\partial \hat{\omega}}{\partial \hat{x}_i} \cdot \frac{\partial \hat{\omega}}{\partial \hat{x}_j} \right] \]
We can now take the Lagrangian frame to be equal to the Eulerian frame before deformation, such that
\[
\frac{\partial \tilde{\omega}}{\partial \tilde{x}^i} = \frac{\partial \tilde{\omega}}{\partial x^i} \quad \wedge \quad \frac{\partial \tilde{\omega}}{\partial \tilde{x}^j} = \frac{\partial \tilde{\omega}}{\partial x^j}
\]
and we can express \( \tilde{\omega} \) in coordinate form
\[
\tilde{\omega} = w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3
\]
Assuming that the initial system is orthonormal i.e. that \( \hat{e}_i \cdot \hat{e}_j = \delta_{ij} \) they are
\[
\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} + \sum_k \frac{\partial w_k}{\partial x_i} \frac{\partial w_k}{\partial x_j} \right]
\]
and for an infinitesimal deformation,
\[
\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right]
\]
Clearly, if a body is simply displaced without deformation, they \( \tilde{\omega} \) is independent of \( x_1, x_2, x_3 \) and hence \( \varepsilon_{ij} \neq 0 \). The tensor \( \varepsilon_{ij} \) describes only the deformation of the body.
Example

Let's calculate the components of the deformation tensor for a deformation described by

\[ x'_1 = x_1 + k x_2^2, \quad k \ll 1 \]
\[ x'_2 = x_2 \]
\[ x'_3 = x_3 \]

Clearly we can put this in the form \( \tilde{\mathbf{e}}' = \tilde{\mathbf{e}} + \tilde{\mathbf{w}} \) with

\[ \tilde{\mathbf{w}} = (k x_2^2, 0, 0) \]

The only non-zero derivative is

\[ \frac{\partial \tilde{\mathbf{w}}}{\partial x_2} = 2k x_2 \]

and hence the only non-zero component of the deformation tensor \( \tilde{\mathbf{e}}' \)

\[ \tilde{\varepsilon}_{12} = \tilde{\varepsilon}_{21} = k x_2 \]

i.e.

\[ \tilde{\varepsilon}_{ij} = \begin{pmatrix} 0 & k x_2 & 0 \\ k x_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
Note that not all elements of the deformation tensor can be independent. First of all, $\varepsilon_{ij}$ is a rank-2 symmetric tensor and hence can have at most six independent elements. However, these elements are calculated from relations of the form

$$\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right]$$

and hence depend only on the three functions $w_1, w_2, w_3$. As a result, only three of the elements of $\varepsilon_{ij}$ can be independent. Indeed, the elements of $\varepsilon_{ij}$ must obey a number of compatibility relations.

For example, let's assume that the functions $w_i$ are continuous with well-behaved second derivatives. Then, by their definition,

$$\varepsilon_{11} = \frac{\partial w_1}{\partial x_1} \quad \text{and} \quad \varepsilon_{22} = \frac{\partial w_2}{\partial x_2}$$

so that

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} = \frac{\partial^3 w_1}{\partial x_2^3} \quad \text{and} \quad \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 w_2}{\partial x_1^3}$$

Adding these two together we obtain

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 w_1}{\partial x_2^3} + \frac{\partial^3 w_2}{\partial x_1^3}$$
\[ \frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial}{\partial x_1} \left[ \frac{\partial \varepsilon_{11}}{\partial x_2} + \frac{\partial \varepsilon_{22}}{\partial x_1} \right] \]

\[ \Rightarrow \frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \]

Similarly one can prove two more conditions:

\[ \frac{\partial^2 \varepsilon_{22}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2 \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} \]

\[ \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{31}}{\partial x_1 \partial x_3} \]

and also conditions of the form

\[ \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_2} \left[ -\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right] \text{ etc.} \]

For a tensor to be a deformation tensor, its components must satisfy all these conditions.