Lecture #30

Fluid Mechanics

The continuity equation

In the next few lectures we will derive the equations that govern the behavior of fluids. We will allow each particle in the fluid to trace its own trajectory, but we will neglect interactions between particles.

We can describe the fluid using the particle distribution function in phase space

\[ f(\mathbf{x}, \mathbf{p}, t) \, d^3x \, d^3p \]

which measures the number of particles (per unit coordinate volume \( d^3x \) and per unit momentum volume \( d^3p \)) which are at coordinates described by the vector \( \mathbf{x} \) and have momenta \( \mathbf{p} \).

Because we neglect interactions between particles, the distribution function is conserved along trajectories of volume elements in phase space, i.e.

\[ \frac{\partial f}{\partial t} + \mathbf{x}_i \frac{\partial f}{\partial \mathbf{x}_i} + \mathbf{p}_i \frac{\partial f}{\partial \mathbf{p}_i} = 0 \]

This equation is called the collisionless Boltzmann or Vlasov equation.
We can also write equation 1 in terms of the velocity

\[ \dot{w}_i = \dot{x}_i. \]

Using mass \( p_i = m \dot{w}_i \) and acceleration \( p_i = ma_i \), equation 1 becomes

\[ \frac{\partial f}{\partial t} + \dot{w}_i \frac{\partial f}{\partial \dot{w}_i} + a_i \frac{\partial f}{\partial w_i} = 0 \]  \( \square \)

When the interparticle distance is comparable to or larger than any characteristic scale in the problem, then we have to solve equation 2, which is very hard. However, when the interparticle distance is very small, they we can make the usual assumption of continuum mechanics and write equations for average quantities such as

the density \[ \rho(x, t) = \int f(x, \dot{w}) d\dot{w} \]

the mean velocity \[ u_i(x, t) = \frac{1}{\rho} \int f(x, \dot{w}) \dot{w}_i d\dot{w} \]

e tc.
Since the mean quantities involve integrals of the distribution function, we will take the integral over $d\tilde{w}$ of equation (2)

$$\int \frac{df}{dt} d^3\tilde{w} + \int \omega_i \frac{df}{dx_i} d^3\tilde{w} + \int a_i \frac{df}{dw_i} d^3\tilde{w} = 0 \quad (3)$$

But:

$$A = \int \frac{df}{dt} d^3\tilde{w} = \frac{\partial}{\partial t} \int f d^3\tilde{w} = \frac{df}{dt}$$

$$B = \int \omega_i \frac{df}{dx_i} d^3\tilde{w} = \frac{\partial}{\partial x_i} \int \omega_i f d^3\tilde{w} = \frac{\partial}{\partial x_i} (\rho u_i)$$

$$C = \int a_i \frac{df}{dw_i} d^3\tilde{w} = \frac{\partial}{\partial w_i} \int (a_i f) d^3\tilde{w} - \left( \int \frac{\partial a_i}{\partial w_i} d^3\tilde{w} \right)$$

$$C_1 = \int \frac{\partial}{\partial w_i} (a_i f) d^3\tilde{w} = \int \nabla (\tilde{a} f) d^3\tilde{w} - \int_{\Sigma} \nabla (\tilde{a} f) d^3\tilde{s}$$

where $d\tilde{s}$ is a surface that covers the volume $d\tilde{w}$. We assume that there is no acceleration at the outermost edges of the velocity volume and hence

$$C_1 = 0.$$
Regarding $C_9$, we will also assume that the acceleration produced by any external force satisfies

$$\frac{\partial \dot{x}_i}{\partial x_i} = \phi$$

hence $C_9 = \phi$.

Putting all back into equation (3) we obtain

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i} (p \dot{u}_i) = \phi$$

In vector form

$$\frac{\partial p}{\partial t} + \nabla (p \dot{u}) = \phi$$

The continuity equation.

The continuity equation can also be written as

$$\frac{\partial p}{\partial t} + p \nabla \cdot \dot{u} + (\nabla p) \cdot \dot{u} = \phi \Rightarrow \frac{\partial p}{\partial t} + \dot{u} \cdot \nabla p + p (\nabla \cdot \dot{u}) = \phi \Rightarrow$$

$$\Rightarrow \frac{\partial p}{\partial t} + p \nabla \cdot \dot{u} = \phi$$

Lagrangian derivative.
The continuity equation describes conservation of mass because:

\[ \frac{\partial p}{\partial t} + \nabla (p \vec{v}) = 0 \Rightarrow \int \frac{\partial p}{\partial t} \, d^3 \vec{x} + \int \nabla (p \vec{v}) \, d^3 \vec{x} = 0 \Rightarrow \]

\[ \Rightarrow \frac{\partial}{\partial t} \int p \, d^3 \vec{x} + \int p \vec{v} \cdot d\vec{s} = \Phi \]

where \( d\vec{s} \) is the surface that covers the volume element \( dV = d^3 \vec{x} \). If we set

\[ dm = \int p \, d^3 \vec{x} \]

the mass enclosed in the volume element \( V = d^3 \vec{x} \), then

\[ \frac{\partial m}{\partial t} + \int (p \vec{v}) \cdot d\vec{s} = 0 \Rightarrow \]

\[ \Rightarrow \frac{\partial m}{\partial t} = - \int (p \vec{v}) \, d\vec{s} \quad (5) \]

But the last term describes the mass flux through the surface \( d\vec{s} \) (and hence out of the volume \( d^3 \vec{x} \)). As a result, equation (5) shows that mass can change in a volume element only if it flows out of or in to the volume element and hence there is no sink or source of matter.
Terminology often used:

1. We call a fluid flow \textit{steady-state} when all the Eulerian derivatives are equal to zero, i.e.,
   \[
   \frac{Dp}{dt} = 0
   \]
   for the continuity equation.

Note that the density of any volume element may change in time along its trajectory, i.e., \( \frac{Dp}{dt} \) may be non-zero, but every snapshot of the flow will always look the same.

2. We call a flow \textit{incompressible} when
   \[
   \frac{Dp}{dt} = 0
   \]
   i.e., from the Lagrangian form of the continuity equation, when
   \[
   \vec{V} \cdot \vec{v} = 0
   \]

3. The continuity equation is often called the \textit{zeroth moment} of the Boltzmann equation.
An example

Let's consider the steady-state wind from a star like our sun. If we study this wind far away from the star it will often look quasispherical with a constant (in magnitude) outflowing velocity $v_0$.

Applying the continuity equation in this case, we obtain

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{u}) = 0 \Rightarrow
\]

\[
\Rightarrow \frac{1}{r^2} \left( \frac{d}{dr} \left[ r^2 \rho v_0 \right] \right) = 0 \Rightarrow r^2 \rho v_0 = \text{constant}.
\]

The value of this constant can be evaluated by noting that the rate of mass loss $\dot{M}$ is equal to

\[
\dot{M} = 4\pi r^2 \rho v_0 = \text{constant}
\]

and hence the density profile of such a wind is

\[
\rho = \frac{\dot{M}}{4\pi r^2 v_0} \sim \frac{1}{r^2}
\]