Lecture #31

Euler's equation

In the previous lecture we started from the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + a_i \frac{\partial f}{\partial u_i} = \phi$$  \(1\)

and derived the continuity equation

$$\frac{\partial p}{\partial t} + \nabla \cdot (p \bar{u}) = \phi$$  \(2\)

by integrating eq (1) over velocity space. Equation (2) is therefore the zeroth-order moment of equation (1) and involves the zeroth-order moment of \(f\), which is the density \(p\), and the first-order moment of \(f\), which is the average velocity \(\bar{u}\).

Now we will take the first-order moment of equation (1). We will multiply eq (1) by \(p_j = m u_j\) and then integrate over \(u_j\).

We will also allow now for the possibility of interactions such that equation (1) becomes

$$\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + a_i \frac{\partial f}{\partial u_i} = \left[ \frac{\partial f}{\partial t} \right]_{\text{inter}}$$  \(3\)
\[ \frac{m w_j}{\partial t} + m w_i w_j \frac{\partial f}{\partial x_i} + m w_j \alpha_i \frac{\partial f}{\partial w_i} = \left[ m \omega_i \frac{\partial f}{\partial t} \right]_{\text{inter}} \]

\[ m \int w_j \frac{\partial f}{\partial t} d^3w = m \int w_i \omega_j \frac{\partial f}{\partial x_i} d^3w + m \int w_j \alpha_i \omega_i \frac{\partial f}{\partial w_i} d^3w = \left\{ \left[ m \omega_i \frac{\partial f}{\partial t} \right]_{\text{inter}} \right\} \]

\[ A = m \int w_j \frac{\partial f}{\partial t} d^3w = \frac{\partial}{\partial t} \left[ m \int w_j f d^3w \right] = \frac{\partial}{\partial t} (\rho w_j) \]

\[ B = m \int w_i w_j \frac{\partial f}{\partial x_i} d^3w = \frac{\partial}{\partial x_i} \left[ m \int w_i w_j f d^3w \right] = \frac{\partial}{\partial x_i} \left[ m \int (w_i - u_i)(w_j - u_j) f d^3w \right] = \frac{\partial}{\partial x_i} \left[ m \int (w_i - u_i) w_j f d^3w + \frac{\partial}{\partial x_i} m \int (w_i - u_i) u_j f d^3w + \frac{\partial}{\partial x_i} m \int u_i u_j f d^3w \right] \]
\[ B = m \frac{d}{dx_i} \int \left( (w_i - u_i)(w_j - u_j) \right) d^2w + \]
\[ + \frac{\partial}{\partial x_i} \left[ \int \left( \frac{\partial f}{\partial w_i} \right) d^2w \right] u_i \]
\[ + m \frac{\partial}{\partial x_i} \int f d^2w \Rightarrow \]
\[ \Rightarrow B = \frac{\partial}{\partial x_i} \left[ m \int \left( (w_i - u_i)(w_j - u_j) \right) f d^2w + pu_i u_j \right] \Rightarrow \]
\[ \Rightarrow B = \frac{\partial}{\partial x_i} \left( \Psi_{ij} + pu_i u_j \right) \]

where we have defined the symmetric, rank-2 tensor
\[ \Psi_{ij} = m \int \left( (w_i - u_i)(w_j - u_j) \right) f d^2w \]

Note that \( w_i - u_i \) represents the "random" velocity of a particle in the direction \( i \).

Continuing with term \( C' \)
\[ C' = m \int w_i a_i \frac{\partial f}{\partial w_i} d^3w = m \int \frac{\partial}{\partial w_i} (w_j a_i f) d^3w - m \int w_i \frac{\partial a_i}{\partial w_i} f d^3w \]
\[ - m \int a_i \frac{\partial w_i}{\partial w_i} f d^3w = \phi - \phi - a_i \delta_{ij} \rho = - \rho a_j \]
For the same reasons as in the derivation of the continuity equation.

Finally,

\[ D = m \left[ \omega_j \frac{\partial}{\partial t} \omega_j + \int \frac{\partial}{\partial t} \left( m \omega_j \right) d\omega \right]_{\text{inter}} = \int \frac{\partial}{\partial t} \left( m \omega_j \right) d\omega = 0 \]

because the average change of momentum during the integral collisions (or interactions) has to be equal to zero. Putting all the terms together, (and setting again)

\[ \frac{\partial}{\partial t} (pu_j) + \frac{\partial}{\partial x_i} (\psi_{ij} + pu_i u_j) - p a_j = 0 \rightarrow \]

\[ \Rightarrow u_i \frac{\partial p}{\partial t} + p \frac{\partial u_i}{\partial t} + \frac{\partial \psi_{ij}}{\partial x_i} + u_i u_j \frac{\partial p}{\partial x_i} + p u_j \frac{\partial u_i}{\partial x_i} + p u_i \frac{\partial u_j}{\partial x_i} - p a_j = 0 \]

\[ \Rightarrow u_j \left[ \frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + p \frac{\partial u_i}{\partial x_i} \right] + p \frac{\partial u_i}{\partial t} + p u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial \psi_{ij}}{\partial x_i} - p a_j = 0 \]

\[ \Rightarrow u_i \left[ \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i} (pu_i) \right] + p \left[ \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} \right] - p a_j + \frac{\partial \psi_{ij}}{\partial x_i} = 0 \]

\( \blacksquare \) from continuity equation
and hence the equation becomes

\[ p \frac{Du_i}{dt} = p a_j - \frac{\partial \phi}{\partial x_i} \]

Euler’s equation

The first two terms are nothing but Newton’s 2nd law.

What is \( \psi_{ij} \), the stress tensor?

1. Diagonal terms

\[ \psi_{ii} = m \int (w_i - u_i)(w_i - u_i) \, d^3w = \rho \langle (w_i - u_i)^2 \rangle \]

i.e., the term \( \psi_{ii} \) is equal to the average square of the random velocities in the i-direction, we call it

If the random velocities are due to thermal motions, we call this the pressure of the fluid.

Recall that \( P = \rho \frac{kT}{m} = \rho \frac{K}{m} = \rho \langle (w - u)^2 \rangle \)

is the definition of pressure in kinetic theory.
If we assume that the off-diagonal terms are zero and that all components of pressure are equal, then Euler's equation becomes

$$p \frac{Du_j}{dt} = -p a_j - \frac{\partial P}{\partial x_j}$$

which shows that a pressure gradient $\frac{\partial P}{\partial x_j}$ causes the velocities of volume elements. This is how winds are formed in the atmosphere!

(i) Off-diagonal terms.

Usually we say that $\psi_{ij}$ describes the flux of i-momentum along the j-direction.

We usually write by subtracting the pressure $P$,

$$\psi_{ij} = P \delta_{ij} - \sigma_{ij}$$

and refer to $\sigma_{ij}$ as the viscosity-stress tensor.

Clearly, $\sigma_{ij}$ is non-zero only if we have a gradient in velocity (otherwise there is no friction). Moreover, $\sigma_{ij}$ must vanish for a purely circular motion

$$\vec{u} = \vec{\omega} \times \vec{v}$$
The most general, rank-2 tensor satisfying these requirements is usually written in the form

\[ \sigma_{ij} = \eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \frac{1}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \].

Note that if \( J = 0 \), the diagonal terms of \( \sigma_{ij} = 0 \).

The parameter \( \eta \) is called the coefficient of shear viscosity.

The parameter \( J \) is called the coefficient of bulk viscosity.