Problem 1: The vector potential due to a source $\mathbf{J}(\mathbf{x}, t)$ may be written

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c} \int d^3x' \int dt' \frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta(t' - t + \frac{|\mathbf{x} - \mathbf{x}'|}{c}).$$

Assume that the source is contained within a region of linear size $\sim L$. Show that at field points $|\mathbf{x}| \gg L$, the Fourier transform of $\mathbf{A}$ may be expanded as

$$\mathbf{A}_\omega(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{c|\mathbf{x}|} \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbf{J}_\omega(\mathbf{x}'') (-i k \frac{\mathbf{x} \cdot \mathbf{x}''}{|\mathbf{x}''|})^n d^3x',$$

where $\mathbf{J}_\omega(\mathbf{x})$ is the corresponding Fourier Transform of $\mathbf{J}(\mathbf{x}, t)$, and $k \equiv \omega/c$. Assume $kL \ll 1$.

Problem 2: Consider a charge $q$ moving around a circle of radius $r_0$ at frequency $\omega_0$. By consideration of the current density and its Fourier Transform, show that the Fourier Transform of the vector potential $\mathbf{A}_\omega(\mathbf{x})$ is nonzero only at $\omega = \omega_0$ in the dipole approximation, nonzero only at $\omega = 2\omega_0$ in the quadrupole approximation and so on. You may want to use the results of Problem 1 above—but be careful in your choice of multipole components!

Problem 3: An almost spherical surface, defined by

$$R(\theta) = R_0 [1 + \beta P_2(\cos \theta)],$$

has inside of it a uniform volume distribution of charge totaling $Q$. The small parameter $\beta$ varies harmonically in time at frequency $\omega$. This corresponds to surface waves on a sphere. Keeping only lowest-order terms in $\beta$ and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated.

Problem 4: Jackson 13.5