Problem 1: In lecture, we found that the potential from a point charge could be written

\[ A^\mu(x) = 2q \int d^4x' \theta(x_0 - x'_0) \delta[(x - x')^2] \int d\tau U^\mu(\tau) \delta^{(4)}(x' - x_q(\tau)). \]

Show that in terms of the scalar potential \( \Phi \) and vector potential \( \vec{A} \), this reduces to

\[ \Phi(\vec{x}, t) = \left[ \frac{q}{(1 - \vec{\beta} \cdot \hat{n})|\vec{x} - \vec{x}_q|} \right]_{\text{ret}}, \]

and

\[ \vec{A}(\vec{x}, t) = \left[ \frac{q\vec{\beta}}{(1 - \vec{\beta} \cdot \hat{n})|\vec{x} - \vec{x}_q|} \right]_{\text{ret}}, \]

where “ret” means the quantity in the brackets is to be evaluated at the retarded time \( \tau_0 \), such that

\[ (x_0 - x_q(\tau_0))^2 = |\vec{x} - \vec{x}_q(\tau_0)|^2. \]

Problem 2: In this problem, we will use relativistic transformations to find the radiation emitted by a particle moving at relativistic speeds.

(a) Show that the total emitted power is a Lorentz invariant for any emitter that emits with front-back symmetry in its instantaneous rest frame.

(b) Show that the covariant generalization of the Larmor formula is

\[ P = \frac{2q^2}{3c^3} a^\alpha a_\alpha, \]

where \( a^\alpha \equiv dU^\alpha/d\tau \) and \( U^\alpha \) is the 4-velocity. Be careful about which frame you’re in!

(c) Show that in terms of the 3-vector acceleration \( d^2\vec{x}/dt^2 \), this power is

\[ P = \frac{2q^2}{3c^3} \gamma^4 (a_\parallel^2 + \gamma^2 a_\perp^2), \]

where \( a_\parallel \) and \( a_\perp \) are the components of acceleration parallel and perpendicular to the direction of \( v \).
Problem 3: Suppose a particle of mass $m$ and charge $q$ is moving (relativistically) in a uniform magnetic field $\vec{B}$.

(a) Show that the total emitted power may be written

$$P = 2\sigma_T c^2 \beta^2 \gamma^2 U_B,$$

where $\sigma_T$ is the Thomson cross section, $\beta = v/c$, and $U_B$ is the magnetic energy density.

(b) Assuming an observer sees the radiation coming only within the cone of half-angle $1/\gamma$ about the velocity vector, show that the duration of the observed pulse once every gyration period is

$$\Delta t \approx \frac{2}{\gamma \omega_B \sin \alpha} \left(1 - \frac{v}{c}\right),$$

where $\omega_B$ is the angular velocity of gyration, and $\alpha$ is the pitch angle, i.e., the angle between the velocity vector and the magnetic field.

Problem 4: A particle is accelerated by a force having components $F_\parallel$ and $F_\perp$ with respect to the particle’s velocity. Show that the radiated power is

$$P = \left(\frac{2q^2}{3m^2c^5}\right)\left(F_\parallel^2 + \gamma^2 F_\perp^2\right).$$

Thus, the perpendicular component has more effect in producing radiation than the parallel component by a factor $\gamma^2$.

Problem 5: An object emits a blob of material at speed $v$ at an angle $\theta$ to the line-of-sight of a distant observer.

(a) Show that the apparent transverse velocity inferred by the observer is

$$v_{app} = \frac{v \sin \theta}{1 - (v/c) \cos \theta}.$$

(b) Show that $v_{app}$ can exceed $c$; find the angle for which $v_{app}$ is maximum, and show that this maximum is $v_{max} = \gamma v$. 

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