Wave mechanics

Planck: \[ E = h \nu = \frac{h}{\lambda} \omega \]

\( (\omega = 2\pi \nu) \)

"angular frequency"

de Broglie: \[ p = \frac{h}{\lambda} = \frac{h}{\lambda} k \]

\( (k = \frac{2\pi}{\lambda}) \)

"wave number"

In terms of their complementary wave/particle properties, the photon and material particles, such as the electron, are
quite similar. Thus, we (2) will endeavor to write down a wave equation for electrons similar to the E+M wave equation:

\[(\Delta^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{A}(\vec{r}, t) = 0,\]

in free space. Here \(\vec{A}\) is the vector potential. A plane-wave solution has the form:

\[\vec{A}(\vec{r}, t) = \vec{E} e^{i(k \cdot \vec{r} - \omega t)}\]

plugging this into the wave
The equation gives

\[- \frac{\hbar^2}{k^2} + \frac{\omega^2}{c^2} \] \[\vec{E} = 0\]

or \(\omega = c |k|\). For a free particle of mass \(m\), one has (neglecting relativistic effects)

\[E = \frac{\vec{p}^2}{2m}\]

\[\hbar \omega = \frac{\hbar^2 \vec{k}^2}{2m}\]

A plane wave

\[\Psi(\vec{r}, t) = A \ e^{i (\vec{k} \cdot \vec{r} - \omega t)}\]

for such a particle must
satisfy a different type of wave equation. Notice that

\[ D\psi = i\hbar k \psi \]

\[ \frac{2\psi}{\hbar t} = -i\hbar \psi \]

\[ p \psi = \hbar k \psi = \frac{\hbar A}{i} \psi \]

\[ E \psi = \hbar \omega \psi = i\hbar \frac{2\psi}{\hbar t} \]

\[ (E - \frac{p^2}{2m}) \psi = 0 \]

\[ \left[ it \frac{\partial}{\partial t} - \left( \frac{\hbar}{i} D \right)^2 \right] \psi = 0 \]

\[ it \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} D^2 \psi \]
In general, for a particle in an external potential $V(\vec{r},t)$, one has

$$E = \frac{\vec{p}^2}{2m} + V(\vec{r},t).$$

We postulate that the quantum mechanical wave function $\psi(\vec{r},t)$ of a particle obeys the Schrödinger equation

$$i \frac{\partial \psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r},t) + V(\vec{r},t) \psi(\vec{r},t).$$
The energy carried by \( L \)
an E&M wave is proportional
to the time-average of
the square of the field,
and hence to \( \overline{|A|^2} \).

Since \( E = nhv \), this
implies that the \# of
photons is proportional
to \( \overline{|A|^2} \). Indeed, the
probability to find a photon
at a given place and time
is proportional to

\[
P(r, \omega t) \propto \overline{|A(r, \omega t)|^2}.
\]
By analogy, we assert that following Max Born, that the probability to observe a material particle is

\[ P(\vec{r},t) \propto |\Psi(\vec{r},t)|^2 \]

(\text{define } |\Psi|^2)

The total probability should be unity:

\[ 1 = \int d^3r \ |\Psi(\vec{r},t)|^2 \]

If \( \Psi(\vec{r},t) \) satisfies this equation, it is said to be normalized.
Since the Schrödinger equation is linear in \( \psi \), we have the important superposition principle:

If \( \psi_1 \) satisfies Eq. (1) and \( \psi_2 \) satisfies Eq. (1), then so does

\[
\phi(\vec{r}, t) = a \psi_1(\vec{r}, t) + b \psi_2(\vec{r}, t)
\]

Normalization requires

\[
1 = \int d^3r |\phi(\vec{r}, t)|^2 = \left| a \right|^2 \int d^3r |\psi_1(\vec{r}, t)|^2 + \left| b \right|^2 \int d^3r |\psi_2(\vec{r}, t)|^2 + a^*b \int d^3r \psi_1^*(\vec{r}, t) \psi_2(\vec{r}, t) + a b^* \int d^3r \psi_1(\vec{r}, t) \psi_2^*(\vec{r}, t).
\]
If \( \int d^3r \, \gamma_1^* \gamma_2 = 0 \),
then \( \gamma_1 \) and \( \gamma_2 \) are said to be "orthogonal."

In that case, we must have \( |a|^2 + |b|^2 = 1 \).

**Example**

2-slit experiment

\[
\begin{align*}
P_1 &= |a|^2 \\
P_2 &= |b|^2 \\
P_{12} &= |a_1 + a_2|^2 \\
P_{12} &= (\gamma_1^* + \gamma_2^*)(\gamma_1 + \gamma_2) \\
&= \gamma_1^* \gamma_1 + \gamma_1^* \gamma_2 + \gamma_2^* \gamma_1 + \gamma_2^* \gamma_2
\end{align*}
\]
\[ P_{12} = P_1 + P_2 + \Delta P \]

\[ \Delta P = \gamma_1 \gamma_2^* + \gamma_1^* \gamma_2 \]

Suppose

\[ \gamma_1 = A e^{i \mathbf{k} \cdot \mathbf{r}_1} = \sqrt{P_1} e^{i \mathbf{k} \cdot \mathbf{r}_1} \]

\[ \gamma_2 = B e^{i \mathbf{k} \cdot \mathbf{r}_2} = \sqrt{P_2} e^{i \mathbf{k} \cdot \mathbf{r}_2} \]

Then

\[ \Delta P = \sqrt{P_1 P_2} (e^{-i \mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} + e^{i \mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)}) \]

\[ = \sqrt{P_1 P_2} \ 2 \cos[\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \]