1.) A block of mass $M$ is attached to a horizontal ideal spring with force constant $k$. The block is resting on a frictionless table, with the spring at its natural length. The block is then struck by a bullet of mass $m$ moving with velocity $v$. The bullet comes to rest inside the block. Find the frequency and amplitude of oscillation of the system after the collision.

First to find the frequency:

After the bullet comes to rest, the system attached to the spring has a mass of $M + m$, so the frequency is:

$$\omega = \sqrt{\frac{k}{M + m}}$$

To find the amplitude, we first use the fact that momentum is conserved when the bullet strikes the block. This means that the velocity of the block just after the bullet comes to rest is:

$$(M + m) v_B = mv$$

$$v_B = \frac{m}{M + m} v$$

After the bullet comes to rest in the block, energy is conserved during oscillation. Initially, the energy is all kinetic (since the spring is at its natural length). When the block reaches its maximal displacement, the energy will be entirely potential. So we can find the amplitude $A$ by equating the energies at these points:

$$\frac{1}{2} (M + m) v_B^2 = \frac{1}{2} kA^2$$

$$A^2 = \frac{m^2 v^2}{k \cdot (M + m)}$$

$$A = mv \sqrt{\frac{1}{k \cdot (M + m)}}$$
2) Most communication satellites are in “geosynchronous” orbit about the Earth, meaning that their orbital period is exactly 1 day (so that they appear to hover above a point on the Earth’s surface). Find the radius of such an orbit. Also, if the satellite has a mass of 100kg, find the additional energy needed for the satellite to escape from the Earth’s gravity. The gravitational constant is \( G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \), and the mass of the Earth is \( 6 \times 10^{24} \ \text{kg} \).

Assume the satellite is in a circular orbit of radius \( r \) from the center of the Earth. The gravitational force on it is:

\[
F = \frac{G M_E m}{r^2}
\]

We can find the velocity of the satellite by equating this with the centripetal force needed to accelerate the satellite around its circle:

\[
F = \frac{m v^2}{r} = \frac{G M_E m}{r^2}
\]

\[
v^2 = \frac{G M_E}{r}
\]

The period of the orbit also depends on \( v \) and \( r \):

\[
T = \frac{2 \pi r}{v} = 2 \pi r \sqrt{\frac{r}{G M_E}}
\]

Since we know we want a period of 1 day (or \( 8.6 \times 10^4 \) s), we can find \( r \):

\[
T = 2 \pi r \sqrt{\frac{r}{G M_E}} = 2 \pi r^{3/2} \sqrt{G M_E}
\]

\[
r = \left[ \frac{T}{2 \pi} \sqrt{G M_E} \right]^{2/3} = \left[ \frac{8.6 \times 10^4 \text{s} \sqrt{6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \cdot 6 \times 10^{24} \text{kg}}}{2 \pi} \right]^{2/3} = 4.2 \times 10^7 \text{m} = 42,000\text{km}
\]

The total energy of this orbit (taking the gravitational potential energy to be zero at infinite distance) is:
\[ E = KE + U = \frac{1}{2} mv^2 - \frac{GM_Em}{r} \]
\[ = \frac{1}{2} m \cdot \frac{GM_E}{r} - \frac{GM_Em}{r} = -\frac{1}{2} \frac{GM_Em}{r} \]
\[ = -\frac{1}{2} \frac{6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \cdot 6 \times 10^{24} \text{ kg} \cdot 100 \text{ kg}}{4.2 \times 10^7 \text{ m}} = -4.8 \times 10^8 \text{ J} \]

Since the satellite will need a total energy of 0J if it is to escape the Earth’s gravity, 4.8 \times 10^8 \text{ J} must be added.

3) A uniform 100\text{ kg} disk of radius 1\text{ m} starts at rest, and is turned by a motor that delivers a time-dependent torque about the disk’s center of mass: \( \tau(t) = 20 \text{ Nm/s} \cdot t \). What is the disk’s angular velocity 10s after the motor is turned on?

The angular acceleration of the disk about its center of mass is given by:

\[ \alpha(t) = \frac{\tau(t)}{I} = \frac{20 \text{ Nm/s} \cdot t}{\frac{1}{2} \cdot 100 \text{ kg} \cdot (1 \text{ m})^2} = 0.4/s^3 \cdot t \]

To find the change in angular velocity we integrate:

\[ \omega(t) = \int_{t=0}^{t=10s} 0.4/s^3 \cdot t \, dt = 0.4/s^3 \left[ \frac{t^2}{2} \right]_0^{10s} = 20/s \]

4) A car of mass 1000\text{ kg} is traveling at 30\text{ m/s} when a deer jumps in its path 100\text{ m} ahead. What is the minimum coefficient of friction between the tires and the road required for the car to stop before hitting the deer? Assume that it takes the driver 0.75\text{ s} to apply the brakes after the deer jumps.

First we find how far the driver goes before applying the brakes:

\[ d = 30 \text{ m/s} \cdot 0.75\text{ s} = 22.5\text{ m} \]

This leaves 77.5\text{ m} to stop before hitting the deer. We can find the acceleration needed from:
\[ v_f = v_i - at = 0 \]
\[ t = \frac{v_i}{a} \]
\[ x = x_i + v_i t - \frac{1}{2} at^2 = 0 + \frac{v_i^2}{a} - \frac{1}{2} \frac{v_i^2}{a} = 77.5 \text{m} \]
\[ \frac{1}{2} \left( 30 \text{m/s} \right)^2 = a = 5.8 \text{m/s}^2 \]

The frictional force slowing the car down is given by:

\[ f = \mu N = \mu mg \]

So the acceleration is \( \mu g \). Therefore we can find the \( \mu \) required by:

\[ \mu g = 5.8 \text{m/s}^2 \]
\[ \mu = \frac{5.8 \text{m/s}^2}{9.8 \text{m/s}^2} = 0.59 \]

5) The diagram below shows two masses attached to a uniform thin rod of length \( L \) and mass \( m \). The rod is free to pivot (without friction) about its center. Initially the rod is held at rest in the horizontal position. After the rod is released it begins to rotate. What will its angular velocity be when it is vertical?

We can define the initial height of the two masses to be \( y = 0 \), so that the initial total energy is zero. Since the pivot does no work (it exerts a force at one point only), mechanical energy is conserved at the rod rotates. When the rod is vertical, the energy is:

\[ E_f = KE + U = \frac{1}{2} I \omega^2 + m_1 g \frac{L}{2} - m_2 g \frac{L}{2} = E_i = 0 \]
\[ \omega^2 = \frac{gL}{I} (m_1 - m_2) \]

The moment of inertia has contributions from the rod and the masses:

\[ I = \frac{m_r L^2}{12} + m_1 \left( \frac{L}{2} \right)^2 + m_2 \left( \frac{L}{2} \right)^2 = \frac{L^2}{12} (m_r + 3m_1 + 3m_2) \]
Therefore $\omega$ is given by:

$$\omega = \sqrt{\frac{gL}{I}} (m_1 - m_2) = \sqrt{\frac{gL}{\frac{L^2}{12} (m_r + 3m_1 + 3m_2)}} (m_1 - m_2) = \sqrt{\frac{12g (m_1 - m_2)}{L (m_r + 3m_1 + 3m_2)}}$$

6) A crane consists of a uniform horizontal beam of length 5m and mass 1000kg attached to two supports, as shown below. If the crane is holding a mass of 5000kg, what force must be exerted by each support to hold the beam in equilibrium?

![Diagram of crane with beam and supports](image)

We take the origin to be the point where the leftmost support holds the beam. Since the crane is in equilibrium, the net torque about this point must be 0:

$$\tau = T_2 \cdot 1m - m_s g \cdot 2.5m - Mg \cdot 5m = 0$$

$$T_2 = \frac{1000kg \cdot 9.8m/s \cdot 2.5m + 5000kg \cdot 9.8m/s \cdot 5m}{1m} = 269.5kN$$

This is the (upwards) force exerted by the right support.

To find the force exerted by the left support, we use the fact that the total vertical force must be 0:

$$T_1 + T_2 - m_s g - Mg = 0$$

$$T_1 = 1000kg \cdot 9.8m/s^2 + 5000kg \cdot 9.8m/s^2 - 73.5kN = -210.7kN$$

Note that this force is downward!

7) The uniform cylinder below is mounted on a frictionless axle through its center, perpendicular to the page. The cylinder has a mass $M$ and radius $R$. A piece of putty of mass $m$ is thrown horizontally with velocity $v$ and sticks to a small tab (of negligible mass) at the bottom edge of the cylinder. What is the angular velocity of the cylinder after the collision?

![Diagram of cylinder with putty](image)
We can solve this problem by realizing that the angular momentum of the cylinder+putty system about the cylinder’s center of mass is conserved. Just before the collision, the angular momentum is carried by the putty alone, and has magnitude:

\[ L_i = mvR \]

After the collision, the angular momentum will be the sum of that from the cylinder and from the putty:

\[ L_f = mv'R + I \omega = mv'R + \frac{1}{2} MR^2 \omega \]

Since the putty is stuck to the outer edge of the rotation cylinder, we have \( v' = \omega R \), so:

\[ L_f = m\omega R \cdot R + \frac{1}{2} MR^2 \omega = \omega R^2 \left( m + \frac{M}{2} \right) = L_i = mvR \]

\[ \omega = \frac{mv}{\left( m + \frac{M}{2} \right) R} \]

8) Two identical stunt cars of mass 1500kg are connected by a 5m long massless cable. They then enter a circular curve that is covered with ice (so that friction between the cars and the road is negligible) and banked at an angle of 20°. Both cars have the same angular velocity with respect to the center of the curve, and they stay at a constant height on the curve. For the car closest to the bottom, the curve has a radius of 100m. What is the tension in the cable?
We begin by defining a coordinate system and identifying all the forces acting on each car. Since in this case the cars are accelerating horizontally, we draw the axes as follows:

**Car 1:**
- $\theta$
- $T$
- $N_1$
- $mg$
- $x$
- $y$

**Car 2:**
- $T$
- $N_2$
- $mg$
- $x$
- $y$

The net force on car 1 is:

$$F_{1,x} = T \cos \theta - N_1 \sin \theta = -\frac{mv^2}{r_1} = -\frac{m(\omega r_1)^2}{r_1} = -m\omega^2 r_1$$
$$F_{1,y} = T \sin \theta + N_1 \cos \theta - mg = 0$$

Note that the $x$ acceleration is negative (to the left). Similarly the net force on car 2 is:

$$F_{2,x} = -T \cos \theta - N_2 \sin \theta = -m\omega^2 r_2$$
$$F_{2,y} = -T \sin \theta + N_2 \cos \theta - mg = 0$$

We can now solve for $N_1$ and $N_2$:

$$T \sin \theta + N_1 \cos \theta - mg = 0$$
$$N_1 = \frac{mg - T \sin \theta}{\cos \theta}$$
$$-T \sin \theta + N_2 \cos \theta - mg = 0$$
$$N_2 = \frac{mg + T \sin \theta}{\cos \theta}$$

Next we plug these values into the equations for the $x$ component of the force:
\[T \cos \theta - \left( \frac{mg - T \sin \theta}{\cos \theta} \right) \sin \theta = -m\omega^2 r_1\]

\[T \left( \cos \theta + \frac{\sin^2 \theta}{\cos \theta} \right) = T \left( \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta} \right) = \frac{T}{\cos \theta} = -m\omega^2 r_1 + mg \tan \theta\]

\[-T \cos \theta - \left( \frac{mg + T \sin \theta}{\cos \theta} \right) \sin \theta = -m\omega^2 r_2\]

\[T \left( \cos \theta + \frac{\sin^2 \theta}{\cos \theta} \right) = \frac{T}{\cos \theta} = m\omega^2 r_2 - mg \tan \theta\]

We now have two equations for \( T / \cos \theta \), so they must be equal:

\[-m\omega^2 r_1 + mg \tan \theta = m\omega^2 r_2 - mg \tan \theta\]

\[2g \tan \theta = \omega^2 (r_1 + r_2)\]

\[\omega^2 = \frac{2g \tan \theta}{r_1 + r_2}\]

We can now solve for \( T \):

\[T = \left( -m\omega^2 r_1 + mg \tan \theta \right) \cos \theta = \left( -m \frac{2g \tan \theta}{r_1 + r_2} - r_1 + mg \tan \theta \right) \cos \theta\]

\[= \frac{-2mr_1 g \sin \theta + mr_2 g \sin \theta + mr_1 g \sin \theta}{r_1 + r_2} = \frac{mg \sin \theta \frac{r_2 - r_1}{r_2 + r_1}}\]

\[= 1500 \text{kg} \cdot 9.8 \text{m/s}^2 \sin (20^\circ) \frac{100 \text{m} + 5 \text{m} \cdot \cos (20^\circ) - 100 \text{m}}{100 \text{m} + 5 \text{m} \cdot \cos (20^\circ) + 100 \text{m}} = 115 \text{N}\]

9) A particle of mass 2kg is at rest. It is then acted upon by a force that has a constant direction but a magnitude that depends on position:

\[F_x(x) = 10 \text{N} + 30 \text{N/m}^2 \cdot x^2\]

where \( x = 0 \) is the particle’s initial position. Find the particle’s velocity when it is at \( x = 10 \text{m} \)

The work done on the particle is:

\[W = \int \mathbf{F} \cdot d\mathbf{r}\]

Since this problem is entirely one-dimensional, this simplifies to:
\[ W = \int_{0}^{10m} F(x) \, dx = \int_{0}^{10m} (10\text{N} + 30\text{N/m}^2 \cdot x^2) \, dx \]
\[ = \left[ 10\text{Nx} + \frac{30\text{N} \cdot x^3}{3} \right]_{0}^{10m} = 100\text{Nm} + 10000\text{Nm} = 10100\text{Nm} \]

By the work-energy theorem, this equals the change in kinetic energy of the particle:

\[ \Delta KE = \frac{1}{2} m \left( v_f^2 - v_i^2 \right) = \frac{1}{2} m \left( v_f^2 - 0 \right) = 10100\text{Nm} \]
\[ v_f = \sqrt{\frac{2 \cdot 10100\text{Nm}}{2\text{kg}}} = 100.5\text{m/s} \]