Rotational Analogs of Velocity and Acceleration

- We have seen that the angle of rotation \( \theta \) plays the same role that \( x \) played in one-dimensional linear motion.
- We also know that \( \theta \) changes as the object rotates.
  - The rate at which \( \theta \) changes is called the angular velocity, and given the symbol \( \omega \):
    \[
    \omega = \frac{d\theta}{dt}
    \]
  - Note that \( \omega \) can be positive or negative, and that it has dimension of \( T^{-1} \).
• It should come as no surprise that the angular velocity can also change over time.
• Therefore we define an angular acceleration as the rate of change in the angular velocity, and use the symbol $\alpha$ for it:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

• Again, this can be positive or negative.
• Dimension is $T^{-2}$.
Example: Rotation with Constant Angular Acceleration

• Let’s work out the kinematics of rotational motion if $\alpha$ is constant:

$$\alpha = \frac{d\omega}{dt}$$

$$\int \alpha dt = \int d\omega$$

$$\alpha t = \omega - \omega_o$$

$$\omega = \alpha t + \omega_o = \frac{d\theta}{dt}$$

$$\int (\alpha t + \omega_o)dt = \int d\theta$$

$$\frac{1}{2} \alpha t^2 + \omega_o t = \theta - \theta_o$$

$$\theta = \frac{1}{2} \alpha t^2 + \omega_o t + \theta_o$$

Analogous to linear velocity with constant acceleration

Analogous to linear position with constant acceleration
Relationships Between Angular and Linear Variables

• When an object is rotating, every point in it is moving with a linear velocity

• We know that if the object rotates through an angle $\phi$, the linear distance moved is given by:

\[ s = r\phi \]

• It then follows that:

\[ v = \frac{ds}{dt} = r \frac{d\phi}{dt} = r\omega \]

\[ a_T = \frac{dv}{dt} = r \frac{d\omega}{dt} = r\alpha \]

• Note that there is also a radial (or centripetal) acceleration:

\[ a_C = \frac{v^2}{r} = \omega^2 r \]
Vector Nature of Rotational Quantities

• So far we’ve developed a set of rotational kinematic quantities which are analogous to those seen in linear motion:

\[ \phi \leftrightarrow x \]
\[ \omega \leftrightarrow v \]
\[ \alpha \leftrightarrow a \]

• But we know that position, velocity, and acceleration are really vector quantities
  – So we might expect that the same is true for the rotational variables

• And it is – partly!
• Just as we didn’t need vectors to describe one-dimensional motion, we don’t need them to describe rotation about a fixed axis
• But in general the axis of rotation is not constant
  – For example, imagine a football thrown with a slight wobble
• To handle such cases we do need vectors
• But there’s a problem
  – The rotation of an object through an angle $\phi$ can’t be represented as a vector
To see why, consider an object that undergoes two 90° rotations, about different axes:
• So we see that the result of two rotations about different axes can depend on the order in which the rotations are done
  – Since for vectors we know that $\mathbf{A} + \mathbf{B}$ always equals $\mathbf{B} + \mathbf{A}$, rotations cannot be considered vector quantities!
• The situation is better, though, if we consider very small rotations:
• The difference between \( r_1 \) and \( r_2 \) is \( \Delta r \):

\[
\Delta r = |r|(1 - \cos \Delta \phi) \mathbf{i} + |r| \sin \Delta \phi \mathbf{j}
\]

\[
\approx |r|(1 - 1) \mathbf{i} + |r| \Delta \phi \mathbf{j}
\]

• We can therefore define a vector \( \Delta \phi \):

\[
\Delta \phi = \Delta \phi \mathbf{j} = \frac{\Delta r}{|r|} \mathbf{j}
\]

• So small rotations do in fact behave as vectors
  – i.e., the order in which they are done doesn’t matter
• This is important since it allows us to define vectors associated with angular velocity and acceleration:

\[ \omega = \frac{d\phi}{dt} \]

A vector, since \( d\phi \) is a small angle

• The magnitude of \( \omega \) the scalar \( \omega \) we discussed earlier
• To determine the direction of \( \omega \), use the *right-hand rule*:
  – Curl the fingers of your right hand in the same direction as the rotation
  – Your thumb then points along the direction of \( \omega \)
• Once \( \omega \) has been defined, we can also define \( \alpha \):

\[ \alpha = \frac{d\omega}{dt} \]
Vector Cross Product

• There is one additional vector operation that will be important in our discussion of rotational motion

• This is a form of vector multiplication that results in another vector
  – We already know about the dot product, where two vectors are multiplied to give a scalar

• The notation used is:

\[ \mathbf{A} \times \mathbf{B} = \mathbf{C} \]

• The magnitude of \( \mathbf{C} \) is given by

\[ |\mathbf{C}| = |\mathbf{A}| |\mathbf{B}| \sin \phi \]
• We still need to define the direction of $C$
• It is perpendicular to the plane in which vectors $\mathbf{A}$ and $\mathbf{B}$ lie
  – Still need to decide between up and down…
• We again do this with the help of the right-hand rule
  – Begin with the fingers of the right hand pointed along the first vector in the cross product
  – Orient your hand such that your fingers curl toward the second vector
  – Your thumb then points in the direction of $C$
Notes on the Cross-Product

• For parallel vectors, the cross product is 0
• The result depends on the order of the vectors in the product:

\[
\mathbf{A} \times \mathbf{B} = - (\mathbf{B} \times \mathbf{A})
\]

• Applying the cross product to the unit vectors, we find:

\[
\begin{align*}
\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\
\mathbf{j} \times \mathbf{k} &= \mathbf{i} \\
\mathbf{k} \times \mathbf{i} &= \mathbf{j}
\end{align*}
\]
This means that for any two vectors:

\[
\mathbf{A} \times \mathbf{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \times (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k})
\]

\[
= A_x B_y (\mathbf{i} \times \mathbf{j}) + A_x B_z (\mathbf{i} \times \mathbf{k})
\]

\[
+ A_y B_x (\mathbf{j} \times \mathbf{i}) + A_y B_z (\mathbf{j} \times \mathbf{k})
\]

\[
+ A_z B_x (\mathbf{k} \times \mathbf{i}) + A_z B_y (\mathbf{k} \times \mathbf{j})
\]

\[
= A_x B_y \mathbf{k} - A_x B_z \mathbf{j}
\]

\[
- A_y B_x \mathbf{k} + A_y B_z \mathbf{i}
\]

\[
+ A_z B_x \mathbf{j} - A_z B_y \mathbf{i}
\]

\[
= \left( A_y B_z - A_z B_y \right) \mathbf{i} + \left( A_z B_x - A_x B_z \right) \mathbf{j} + \left( A_x B_y - A_y B_x \right) \mathbf{k}
\]