Kinetic Energy of Rotation

• So far we’ve discussed three types of energy
  1. Kinetic energy association with the motion of the center of mass of a system
  2. Potential energy associated with conservative forces acting within a system
  3. Internal energy due to motion with respect to the center of mass, or non-mechanical potential energies (the chemical energy stored in a tank of gas, for example)
• Rotations can be thought of as a special case of internal energy
  – Different parts of the system are moving with respect to the center of mass, but this motion is not at all random, as it is for energy due to heat
• To determine how much kinetic energy a rotating system contains, we first find the value for a single particle moving in a circle:

\[ \text{The kinetic energy is } \frac{1}{2} m v^2 = \frac{1}{2} m (\omega r)^2 \]
For a collection of particles rotating about an axis, then, we would simply sum the kinetic energy of each:

\[ KE = \frac{1}{2} \sum_{i=1}^{N} m_i (\omega r_i)^2 = \frac{1}{2} \omega^2 \sum_{i=1}^{N} m_i r_i^2 \]

The quantity \( \sum_{i=1}^{N} m_i r_i^2 \) is simply the moment of inertia of the system.

Therefore,

\[ KE = \frac{1}{2} I \omega^2 \]

This is exactly analogous to \( KE = \frac{1}{2} mv^2 \) for linear motion.
Example 1: Atwood’s Machine

An Atwood’s machine is initially at rest in the following configuration:

\[ r = 0.049\text{m} \]
\[ I = 0.0173\text{kg-m}^2 \]

\[ m_1 = 0.512\text{kg} \]
\[ m_2 = 463\text{kg} \]

What will the velocity of the heavier block be after it has fallen a distance \( h = 0.765\text{m} \)?
• We take our system to be the Atwood’s machine + Earth
  – Since no external work is done on this system, and all internal forces are conservative, mechanical energy is conserved
  – But the mechanical energy includes energy due to rotation of the pulley!

• We can define the initial energy to be 0. The final energy is then:

\[
E_f = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 + \frac{1}{2} I \omega^2 + m_2 gd - m_1 gd = 0
\]

\[
\frac{1}{2} v^2 (m_1 + m_2) + \frac{1}{2} I \omega^2 + gd (m_2 - m_1) = 0
\]
• Since the string is not slipping on the pulley, we have \( v = \omega r \):

\[
\frac{1}{2} v^2 (m_1 + m_2) + \frac{1}{2} I \left( \frac{v}{r} \right)^2 + gd (m_2 - m_1) = 0
\]

\[
v^2 \left( m_1 + m_2 + \frac{I}{r^2} \right) = gd (m_1 - m_2)
\]

\[
v = \sqrt{\frac{gd (m_1 - m_2)}{m_1 + m_2 + \frac{I}{r^2}}}
\]

• Plugging in the numbers, this gives \( v = 0.21 \text{m/s} \)
Combined Rotational and Translational Energy

- Consider an arbitrary object that is moving (i.e., $v_{cm} \neq 0$) and also rotating about a fixed axis passing through the center of mass:

- The kinetic energy of the part of the object near P is:
  
  \[ KE_n = \frac{1}{2} m_n v_n^2 \]

  so the total KE is:

  \[ \sum_{n=1}^{N} \frac{1}{2} m_n v_n^2 \]

- We can rewrite this by noting that:

  \[ v_n^2 = v_n \cdot v_n = (v_{cm} + v'_n) \cdot (v_{cm} + v'_n) = v_{cm}^2 + 2v_n \cdot v_{cm} + v'_n^2 \]
• The kinetic energy is now:

\[ KE = \sum_{n=1}^{N} \frac{1}{2} m_n \left( v_{cm}^2 + 2v'_n \cdot v_{cm} + v'^2_n \right) \]

• The first term is simply \( \frac{1}{2} Mv_{cm}^2 \)

• The second term is:

\[ \sum_{n=1}^{N} \frac{1}{2} m_n 2v'_n \cdot v_{cm} = v_{cm} \cdot \sum_{n=1}^{N} m_n v'_n = v_{cm} \cdot \frac{v'_{cm}}{M} \]

• Since the primed frame is defined as one in which the center-of-mass is at rest, \( v'_{cm} = 0 \), so this term disappears
• We are left with the third term. Here we need to remember that the object was assumed to be rotating about the center of mass, so \( v'_n = \omega r'_n \).

• Therefore:

\[
\sum_{n=1}^{N} \frac{1}{2} m_n v'_n^2 = \frac{1}{2} m_n v'_n^2 = \frac{1}{2} m_n \left( \omega r'_n \right)^2
\]

\[
= \frac{1}{2} \omega^2 \sum_{n=1}^{N} m_n r'_n
\]

\[
= \frac{1}{2} I_{cm} \omega^2
\]

• So the total kinetic energy is:

\[
KE = \frac{1}{2} Mv_{cm}^2 + \frac{1}{2} I_{cm} \omega^2
\]
Example: Sphere Rolling Down a Ramp

• A sphere starts at rest atop a ramp of height $h$. What is its speed upon reaching the bottom, assuming it rolls without slipping?

• Note that friction must act on the sphere (otherwise it would slide rather than roll)
  – But since it’s not slipping, this friction is static, and therefore does no work
  – Mechanical energy is conserved!
• We can define the initial energy to be 0

\[ E_f = -mg h + \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = E_i = 0 \]

• For a sphere, \( I = \frac{2}{5} m r^2 \). Also, since it’s rolling without slipping, \( \omega = \frac{v}{r} \):

\[-mgh + \frac{1}{2} m v^2 + \frac{1}{2} \left( \frac{2}{5} m r^2 \right) \left( \frac{v}{r} \right)^2 = 0\]

\[ \frac{1}{2} v^2 + \frac{1}{5} v^2 = gh \]

\[ \frac{7}{10} v^2 = gh \]

\[ v = \sqrt{\frac{10}{7} gh} \]