Orbital Motion

• After Copernicus placed the sun, rather than the Earth, at the center of the universe, understanding the motion of planets became simpler
• Johannes Kepler was able to combine centuries of observational data into three laws which governed the motion of all planets:
  1. All planets move in elliptical orbits, with the sun at one focus
  2. The area swept out by a planet’s orbit in a given time interval is constant
  3. The square of the period of any planet’s orbit is proportional to the cube of the average distance from the sun

Newton realized that all these laws could be explained by the physics he had developed
• We’ll start with the third one, and consider a circular orbit
• We know that:

\[
\frac{mv^2}{r} = F = \frac{GM_{\text{sun}} m}{r^2}
\]

\[T = \frac{2\pi r}{v}, \text{ so:} \]

\[
m\left(\frac{2\pi r}{T}\right)^2 = \frac{GM_{\text{sun}} m}{r^2}
\]

\[4\pi^2 r = \frac{GM_{\text{sun}} T^2}{r^2}
\]

\[
\left(\frac{4\pi^2}{GM_{\text{sun}}}\right) r^3 = T^2
\]

Also works for elliptical orbits, but we won’t prove that here
Kepler’s Law of Areas

• Consider a small area “swept out” during an orbit

\[ dA = \frac{1}{2} r \cdot rd\theta \]

\[ \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega \]

• Since the gravitational force exerts no torque, we know that angular momentum is conserved:

\[ L_z = I \omega = mr^2 \omega = \text{const} \]

• This means that:

\[ \frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{L_z}{2m} = \text{const} \]

Implies that velocity is greatest when planet is nearest to the sun
Kepler’s Law of Orbits

- It’s clear that Newton’s Law of Gravity allows perfectly circular orbits.
- To see that elliptical orbits are also reasonable, we need to look at the energy of orbital motion:

\[
E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}
\]

- We can break \( v \) up into components parallel to and perpendicular to the radial direction:
  - i.e., components along \( \hat{r} \) and \( \hat{\theta} \)

\[
v^2 = \mathbf{v} \cdot \mathbf{v} = (v_r \hat{r} + v_\theta \hat{\theta}) \cdot (v_r \hat{r} + v_\theta \hat{\theta})
\]

\[
= v_r^2 + v_\theta^2 = v_r^2 + r^2 \omega^2 = v_r^2 + \frac{(mr^2 \omega)^2}{r^2} = v_r^2 + \frac{L^2}{m^2 r^2}
\]
• Hence the energy is:

\[
E = K + U = \frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}
\]

• If we just want to explore the \textit{radial} motion of the planet, we can write:

\[
E = K_r + U_{\text{eff}}
\]

\[
K_r = \frac{1}{2}mv_r^2
\]

\[
U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r}
\]
• The effective potential looks like:

![Diagram of effective potential with labels]

- Hyperbolic path around planet
- Parabolic path around planet
- Elliptical orbit (planet oscillates between $r_{\text{min}}$, $r_{\text{max}}$)
- Circular orbit (only one value of $r$ allowed)

• To fully prove Kepler’s Law of Orbits, we’d also need to demonstrate that the period of radial oscillations is the same as the orbital period