I: Multiple choice (5 pts. each)

For the multiple-choice questions, please provide a sentence or two explaining why you chose your answer. (A correct answer without explanation will receive 2 pts.)

The first two problems refer to the graph below which shows the potential energy as a function of position, for a system in which mechanical energy is conserved:

1) A particle with total energy of 8J moves in the above potential. At what point does it feel the largest force?

A) Point A  
B) Point B  
C) Point C  
D) Point D

The force is related to the potential energy by \( F = -\frac{dU}{dx} \), so we want the point where \( U \) has the largest derivative, or steepest slope. Among the choices given, point A has the steepest slope. (A)
2) A 10-kg particle is moving in the above potential, and is observed to have zero velocity at point A. Approximately how fast is it moving at point B?

A) The particle can’t reach point B  
B) 10m/s  
C) 5m/s  
D) 1.4 m/s  

Since the particle has 0 velocity at point A, where $U$ is also close to 0, we know its mechanical energy is nearly 0J. At point B, the potential energy is –10J. Since mechanical energy is conserved, we have:

\[ 0J = KE + U = \frac{1}{2}mv^2 - 10J \]

\[ v^2 = \frac{2 \cdot 10J}{10kg} = 2 \frac{m^2}{s^2} \]

\[ v = \sqrt{2} m/s = 1.4m/s \quad (D) \]

3) A steam locomotive has driving wheels with a radius of 1.2m, and under full steam the wheels are observed to rotate at a rate of 6 revolutions per second. What is the speed of the locomotive, assuming the wheels are not slipping on the rails?

A) 7.2 m/s  
B) 22.6 m/s  
C) 45.2 m/s  
D) 99.9 m/s  

For each revolution of the wheels, a point on the edge (where the wheel touches the rail) travels a distance of:

\[ s = 2\pi r = 7.5m \]

So in six revolutions that point moves 45.2m. Since the wheels aren’t slipping on the rails, this is also the linear distance the locomotive moves relative to the rails. Since it happened in 1s, the linear speed is 45.2m/s (C).
4) Consider a one-dimensional system in which a single conservative force acts. The potential energy associated with that force is: \( U(x) = x^3 - 3x \). At which value of \( x \) is the system in an unstable equilibrium?

A) \( x = 0 \)
B) \( x = 0.33 \)
C) \( x = 1 \)
D) \( x = -1 \)

An equilibrium point occurs when \( \frac{dU}{dx} = 0 \). In our case, we want:

\[
\frac{dU}{dx} = 3x^2 - 3 = 0
\]
\[
x = \pm 1
\]

For an unstable equilibrium, \( \frac{d^2U}{dx^2} < 0 \). In our case,

\[
\frac{d^2U}{dx^2} = 6x
\]

so \( x = -1 \) is the only unstable equilibrium point. (D).
II. Problems:

Please show all your work (use the back of the paper if needed)

II.1 (30 pts):

A special spring is constructed that does not obey Hooke’s law; rather the force exerted by the spring is given by:

\[ F(x) = -kx^2 \]

where \( k \) is 1000N/m\(^2\). The spring is standing vertically, and a rock with mass 20kg is placed on top. The rock is then pushed down, compressing the spring by 2.0m. The rock is then released, and the spring launches it into the air. How far does it rise above its initial position?

Since the spring force is conservative, the mechanical energy of the rock+spring+earth system is constant. If we take the rock’s initial height (sitting on the compressed spring) as \( y = 0 \), the initial energy is:

\[ E = U_{spring} \]

To determine the potential energy stored in the spring, we need to find the work that is done to compress it a distance \( d \):

\[ W = \int_{0}^{d} -F(x) \, dx = 1000N/m^2 \int_{0}^{d} x^2 \, dx = \frac{1000N/m^2}{3} d^3 = U(d) \]

So the potential energy stored in the spring is:

\[ U(2.0m) = \frac{1000N/m^2}{3} (2.0m)^3 = 2700J \]

When the block stops rising, the energy is:

\[ E = mgy = 20kg \cdot 9.8m/s^2 \cdot y = 2700J \]

Solving for \( y \) gives:

\[ y = \frac{2700J}{20kg \cdot 9.8m/s^2} = 13.8m \]
2) (25 pts) An F-14 fighter jet has a mass of 20,000 kg. It is observed to be climbing at an angle of 45° at a speed of 300 m/s. How much power are its engines producing at this point? For this problem, assume that air resistance is negligible.

We take the F-14 to be our system. Conservation of energy tells use that:

$$\Delta E = \Delta E_{\text{int}} + \Delta KE + \Delta U + W_{\text{ext}} = 0$$

The velocity is constant, so $\Delta KE = 0$. Since the earth is outside our system, there is no internal potential energy, so $\Delta U = 0$ also. Gravity is an external force, so:

$$W_{\text{ext}} = W_{\text{gravity}} = mg (-\Delta y)$$

Hence the conservation of energy equation becomes:

$$\Delta E_{\text{int}} - mg \Delta y = 0$$

Since we want to determine power, we need to find the rate at which $E_{\text{int}}$ changes:

$$P = \frac{\Delta E_{\text{int}}}{\Delta t} = mg \frac{\Delta y}{\Delta t} = mgv_y = mgv \sin \theta$$

$$= 20000 \text{ kg} \cdot 9.8 \text{ m/s}^2 \cdot 300 \text{ m/s} \cdot \sin 45°$$

$$= 41.6 \times 10^6 \text{ W} = 56000 \text{ hp}$$
3) (25 pts) A disk of radius 1m starts from rest and then undergoes a non-constant angular acceleration of: \( \alpha(t) = 5 \text{radians/s}^3 \). 10 seconds after the disk starts rotating, find:

a) (10 pts) The angular velocity of the disk

We know that:

\[
\alpha = \frac{d\omega}{dt} \\
d\omega = \alpha dt \\
\omega(t) = \int \alpha dt = \int \frac{5 \text{rad}}{s^3} dt = \frac{5 \text{rad}}{s^3} \frac{t^2}{2} + \omega_o
\]

In this problem, \( \omega_o = 0 \), so

\[
\omega(10s) = \frac{5 \text{rad} (10s)^2}{2 s^3} = 250 \text{rad/s}
\]

b) (10 pts) The angle through which it has rotated

To find \( \phi \) we just keep integrating:

\[
\phi(t) = \int \omega dt = \int \frac{5 \text{rad}}{s^3} \frac{t^2}{2} dt = \frac{5 \text{rad}}{s^3} \frac{t^3}{6} + \phi_o
\]

We’re free to choose \( \phi_o = 0 \), so:

\[
\phi(10s) = \frac{5 \text{rad} (10s)^3}{6 s^3} = 833 \text{rad}
\]

c) (5 pts) The tangential acceleration of a point on the edge of the disk.

Tangential and angular acceleration are related by:

\[
a_T = \alpha r
\]

So we have:

\[
a_T = 5(10s) \text{ radians/s}^3 \cdot 1m = 50 \text{m/s}^2
\]