I: Multiple choice (5 pts. each)

For the multiple-choice questions, please provide a sentence or two explaining why you chose your answer. (A correct answer without explanation will receive 2 pts.)

1) What types of systems move according to simple harmonic motion?

A) Only masses attached to ideal springs  
B) Almost any system, near a stable equilibrium point  
C) Any system with periodic motion  
D) Any system acted on by conservative forces

Near a stable equilibrium point $x_o$, one can expand the potential energy as:

$$U(x) = \frac{1}{2} \frac{d^2U}{dx^2} \bigg|_{x_o} \left( x - x_o \right)^2$$

which is just the simple harmonic oscillator potential. So, simple harmonic motion applies to any system near a stable equilibrium point, except for the special case $\frac{d^2U}{dx^2} \bigg|_{x_o} = 0$.

2) The electric fan shown below has blades made of steel rather than the typical plastic, and is spinning very rapidly. The fan is attached to an arm that is free to pivot in all three dimensions about the point shown. If someone pushes down on the end of the arm opposite to the fan blades, what happens to the end of the arm with the fan blades attached?

A) It moves in to the page  
B) It moves out of the page  
C) It moves upwards  
D) It moves downwards
Let’s take the pivot point as the origin. The force shown then exerts a torque (using the right-hand rule) that points out of the page. The initial angular momentum of the blades is to the right (again, a right-hand rule tells us this). Since the blades need to gain a component of angular momentum out of the page in response to the applied torque, the end of the arm upon which they are mounted will turn out of the page. (B).

3) Which of the following equations is always true?

A) \( \vec{L} = I \vec{\omega} \)
B) \( \vec{L} = \text{constant} \)
C) \( \vec{\tau} = I \vec{\alpha} \)
D) \( \vec{\tau} = \frac{d\vec{L}}{dt} \)

Of these, only the last is always true (it’s the rotational version of Newton’s Second Law). (D)
II. Problems:

Please show all your work (use the back of the paper if needed)

1) (35 pts) The pendulum shown below consists of a uniform rod of length $l = 1.0\text{m}$ and mass $m = 1.0\text{kg}$ attached to a uniform disk of radius $r = 0.5\text{m}$ and mass $M = 3.0\text{kg}$.

![Pendulum Diagram]

a) (5 pts) Find the position of the center of mass of the pendulum.

Using the symmetry of the rod and disk to treat them as point particles with center of mass at their geometric center, we find:

$$d_{cm} = \frac{md_{cm,rod} + Md_{cm,disk}}{m + M} = \frac{m \frac{L}{2} + M (L + r)}{m + M}$$

$$= \frac{1.0\text{kg} \cdot 0.5\text{m} + 3.0\text{kg} \cdot (1.5\text{m})}{4.0\text{kg}} = 1.25\text{m}$$

b) (10 pts) Find the moment of inertia of the pendulum about an axis through the pivot point, and perpendicular to the page

To find $I$ for the entire object, we sum the moments of inertia for the disk and the rod (using the parallel axis theorem in the latter case):
\[ I = I_{\text{rod}} + I_{\text{disk}} = \frac{1}{3}mL^2 + \frac{1}{2}Mr^2 + M(L + r)^2 \]
\[ = \left( M + \frac{1}{3}m \right)L^2 + \frac{3}{2}Mr^2 + 2MLr \]
\[ = 3.33\text{kg}(1.0\text{m})^2 + 4.5\text{kg}(0.5\text{m})^2 + 6.0\text{kg} \cdot 1.0\text{m} \cdot 0.5\text{m} \]
\[ = 7.5\text{kg} \cdot \text{m}^2 \]

c) (5 pts) Find the torque exerted by gravity when the pendulum is at an angle \( \theta \) with respect to the vertical.

The torque has a magnitude:
\[ \tau = -(m + M)gd_{cm}\sin \theta \]
where the negative sign is needed since this torque will tend to decrease \( \theta \).

Plugging in what we know about \( d_{cm} \), this becomes:
\[ \tau = -4.0\text{kg} \cdot 9.8\text{m/s}^2 \cdot 1.25\text{m} \cdot \sin \theta = 49.0\text{Nm} \cdot \sin \theta \]

d) (15 pts) Find the frequency of small oscillations of the pendulum.

Since the oscillations are small, we can approximate the torque as:
\[ \tau = -g \left[ m\frac{L}{2} + M(L + r) \right] \theta = -49.0\text{Nm}\theta \]
Which means the equation of motion will be:
\[ \tau = I\alpha = I\frac{d^2 \theta}{dt^2} = -g \left[ m\frac{L}{2} + M(L + r) \right] \theta \]

Plugging in the value of \( I \) from part (b) gives:
\[ 7.5\text{kg} \cdot \text{m}^2 \frac{d^2 \theta}{dt^2} = -49.0\text{Nm}\theta \]
which is of the same form as the equation for simple harmonic motion, with:
"k" = \frac{2\pi^2}{g} \left( \frac{L}{2} + M(L + r) \right) = 49.0 \text{Nm}

"m" = \left[ \left( M + \frac{1}{3}m \right)L^2 + \frac{3}{2}Mr^2 + 2MLr \right] = 7.5 \text{kg} \cdot \text{m}^2

Since for any simple harmonic motion, \( \omega = \sqrt{\frac{k}{m}} \), we just have to substitute in to find:

\[
\omega = \sqrt{\frac{g \left( \frac{L}{2} + M(L + r) \right)}{\left[ \left( M + \frac{1}{3}m \right)L^2 + \frac{3}{2}Mr^2 + 2MLr \right]}} = \sqrt{\frac{49.0 \text{Nm}}{7.5 \text{kg} \cdot \text{m}^2}} = 2.56 / \text{s}
\]
2) (25 pts) The object shown below is a uniform cylinder of radius \( r = 30 \text{ cm} \) and mass \( M = 2.5 \text{ kg} \), with two point masses \( m = 1.0 \text{ kg} \) attached at the outer edge. It starts from rest on a ramp with a height \( h = 0.75 \text{ m} \) and angle with respect to the horizontal \( \theta = 22.5^\circ \), and rolls down without slipping. Find the linear velocity of the object when it reaches the bottom of the ramp.

\[ \text{Since the object rolls without slipping, we can use conservation of energy. The initial energy is entirely potential:} \]
\[ E_i = U = (M + 2m) gh \]

\[ \text{While the final energy consists of both rotational and translational kinetic energy:} \]
\[ E_f = \frac{1}{2} (M + 2m) v^2 + \frac{1}{2} I \omega^2 \]

where \( I \) is calculated about the center of mass of the object. To find it, we simply add up the moments of inertia of the cylinder and the points:
\[ I = \frac{1}{2} Mr^2 + 2mr^2 \]

Finally, since the object is rolling without slipping, we know that \( \omega = \frac{v}{r} \). So,
\[(M + 2m)gh = \frac{1}{2}(M + 2m)v^2 + \frac{1}{2}\left[\frac{1}{2}Mr^2 + 2mr^2\right]\left(\frac{v}{r}\right)^2\]

\[v^2\left[\frac{1}{2}M + m + \frac{1}{4}M + m\right] = (M + 2m)gh\]

\[v^2\left[\frac{3}{4}M + 2m\right] = (M + 2m)gh\]

\[v = \sqrt{(M + 2m)gh} = \sqrt{\frac{(4.5\text{kg}) \cdot 9.8\text{m/s}^2 \cdot 0.75\text{m}}{[1.9\text{kg} + 2.0\text{kg}]}} = 2.9\text{m/s}\]
3) (25 pts) The crane shown below consists of a uniform boom of mass $m$ and length $L$ held by two supports a distance $d$ apart (assume $d < L/2$). Find the force exerted by each support if the crane is holding up a mass $M$ as shown:

We start by identifying all the forces acting on the boom, as shown above. For the crane to be in equilibrium, both the net force and net torque must be zero. Taking the position of the left-hand support as the origin, we have:

$$F_{\text{net}} = -F_1 + F_2 - mg - Mg = 0$$
$$\tau_{\text{net}} = F_2 d - mg \frac{L}{2} - Mg L = 0$$

From the second equation, we see that:

$$F_2 = \frac{g L \left( \frac{m}{2} + M \right)}{d}$$

Plugging this back into the first equation gives:

$$-F_1 + \frac{g L \left( \frac{m}{2} + M \right)}{d} - mg - Mg = 0$$
$$F_1 = \frac{g L \left( \frac{m}{2} + M \right)}{d} - mg - Mg = g \left[ m \left( \frac{L}{2d} - 1 \right) + M \left( \frac{L}{d} - 1 \right) \right]$$