Lecture 15: Systems of Particles

• So far, we’ve considered the motion of *particles*
  – Meaning objects with every part of the object having the same velocity

• We now want to expand our scope to include the study of a group of particles, each of which may have a different velocity
  – This is the more common case: whenever an object is vibrating or rotating, different parts of it have different velocities

• We call such a group of particles a *system*
  – Just like defining a coordinate system, choosing which set of particles form a system is a matter of convenience
A Two-Particle System: Center of Mass

• The simplest case would be a group of two particles:

• In general the free-body diagram for each particle can be summarized by:
• Thus we see that the force on each particle has contributions from interactions with other particles in the system (internal forces) and interactions with the rest of the universe (external forces)

• In other words:

\[ m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 = \mathbf{F}_{1,\text{ext}} + \mathbf{F}_{12} + \mathbf{F}_{2,\text{ext}} + \mathbf{F}_{21} \]

• Newton’s Third Law tells us that \( \mathbf{F}_{12} = -\mathbf{F}_{21} \), so we’re left with:

\[ m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 = \mathbf{F}_{1,\text{ext}} + \mathbf{F}_{2,\text{ext}} = \mathbf{F}_{\text{net,ext}} \]
• Nothing too astounding so far. But, wouldn’t it be nice if we could find a point \( p \) such that:

\[
(m_1 + m_2)\mathbf{a}_p = \mathbf{F}_{\text{net,ext}}
\]

• This magical point moves as though it were a single particle of mass \( m_1 + m_2 \) being acted on by an external force.

• So we want:

\[
(m_1 + m_2)\mathbf{a}_p = m_1\mathbf{a}_1 + m_2\mathbf{a}_2
\]

\[
\mathbf{a}_p = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}
\]
• We can now integrate to find the location of $\mathbf{p}$
  – I’ll drop the constants of integration since I’m interested in getting the simplest answer

$$\mathbf{v}_p = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

$$\mathbf{p} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

• This point is the *center of mass* of our two-particle system
• The motion of each particle is then the sum of the motion of the center of mass (which depends only on forces external to the system) and its velocity relative to the center of mass (which depends only on forces internal to the system)
Many-particle Systems

- We can readily extend this idea to a system of many particles.
- The center of mass is given by

\[
\mathbf{r}_{\text{cm}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \ldots + m_N \mathbf{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i}
\]

\[
= \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{r}_i
\]

- Where \( M \) is the total mass of all the particles in the system.
Solid Object

• We can think of any solid object as a collection of an infinite number of tiny particles, held together by internal forces between them.

• In this case, the sum becomes an integral:

\[ r_{cm} = \frac{1}{M} \int r \, dm \]
Example: Non-particle Projectile

• We’ve learned earlier about the motion of projectile particles
• Now consider the case where an object of arbitrary shape is thrown through the air
  – The object may be vibrating and rotating
  – Thus the motion of any part of the object may be very complex
• However, the external force on the system is simply $mg$, where $m$ is the mass of the entire object
  – We define the object as our system
• Hence the center of mass moves just like a particle projectile
  – That is, in a parabola
Center of mass for two-body system:
Example

• We have a dog on a boat, some distance from the shore:

\[ x_b \quad x_d \]

• The dog then walks a distance \( d \) on the boat toward shore:

\[ x'_d \quad x'_b \]
• We are told that the water is frictionless, and asked to find $x'_b$.
• The key point is that all the horizontal forces in this problem are internal to the dog+boat system. So the center of mass of that system can’t move, so:

$$x_{cm,i} = \frac{m_b x_b + m_d x_d}{m_b + m_d} = x_{cm,f} = \frac{m_b x'_b + m_d x'_d}{m_b + m_d}$$

$$m_b x_b + m_d x_d = m_b x'_b + m_d x'_d$$

• Still have two unknowns and only one equation. But we also know how far the dog moved with respect to the boat, so:

$$x_d - x_b = x'_d - x'_b + d$$

$$x'_d = x_d - x_b + x'_b - d$$
• Substituting in this expression for $x_d'$, we find:

\[ m_b x_b + m_d x_d = m_b x'_b + m_d \left( x_d - x_b + x'_b - d \right) \]

\[ x'_b \left( m_b + m_d \right) = m_d \left( x_b + d \right) + m_b x_b \]

\[ x'_b = \frac{m_d \left( x_b + d \right) + m_b x_b}{m_b + m_d} \]

• Limiting cases:
  - If $d = 0$, $x_b = x'_b$
  - If $m_d = 0$, $x_b = x'_b$
  - If $m_b = 0$, $x'_b = x_b + d$