Lecture 18: Energy

• So far we’ve studied the interaction between objects in terms of the forces between them
• But there another way to think about it: In terms of the exchange of energy between objects
• Energy is somewhat similar to momentum in that it is not a fixed quantity of an object (nor is it the same in every inertial frame)
• As with momentum, there is energy associated with the motion of an object
  – This is called kinetic energy
• But there may also be energy associated with an object’s position
  – This is called potential energy
Vector Dot Product

• Our study of energy will require an additional tool – the multiplication of vectors
• It turns out there are two ways of doing it
  – One gives a *scalar* result, and is denoted $\mathbf{A} \cdot \mathbf{B}$
  – The other gives a *vector* result, and is denoted $\mathbf{A} \times \mathbf{B}$
• We’re interested in the first one, the “dot product”, for now
• Definition:

\[
\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta
\]
Special Cases

- For two parallel vectors:
  \[ \theta = 0 \]
  \[ \cos \theta = 1 \]
  \[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta = |\mathbf{A}| |\mathbf{B}| \]

- For two perpendicular vectors:
  \[ \theta = 90^\circ \]
  \[ \cos \theta = 0 \]
  \[ \mathbf{A} \cdot \mathbf{B} = 0 \]
Algebraic Definition

- If you aren’t given the angle between two vectors, but do know all the components, you can still find the dot product:

\[
A \cdot B = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \cdot (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) = A_x B_x \mathbf{i} \cdot \mathbf{i} + A_x B_y \mathbf{i} \cdot \mathbf{j} + A_x B_z \mathbf{i} \cdot \mathbf{k} + A_y B_x \mathbf{j} \cdot \mathbf{i} + A_y B_y \mathbf{j} \cdot \mathbf{j} + A_y B_z \mathbf{j} \cdot \mathbf{k} + A_z B_x \mathbf{k} \cdot \mathbf{i} + A_z B_y \mathbf{k} \cdot \mathbf{j} + A_z B_z \mathbf{k} \cdot \mathbf{k} = A_x B_x + A_y B_y + A_z B_z
\]
Work

• The concept of work is a familiar one
• We know that as we do work, we get tired
  – In other words, we must expend energy to do work
• In physics, work and energy are indeed related, but to see how we need to be more precise about how these quantities are defined
• In everyday life, “work” might be defined as anything one has to do that isn’t pleasant
• In physics, though, the work done by a force on an object is defined as:

\[ W = \mathbf{F} \cdot \Delta \mathbf{r} \]

where \( \Delta \mathbf{r} \) is the change in the object’s position as the force is acting on it
Examples of work

• Consider first a ball on a string, moving in a circle at constant velocity:

• How much work does the string do?
• None!
  – Although the string accelerates the ball, its force is always perpendicular to the velocity

\[ W = \mathbf{F} \cdot \Delta \mathbf{r} = |\mathbf{F}| |\Delta \mathbf{r}| \cos 90^\circ = 0 \]
Example 2: Lifting an object

• How much work must one do to lift an object (at constant velocity) a height $h$?

• The force one needs to apply is $F = mg$ in the upwards direction, to balance the force of gravity pulling downward.

• If we pull directly upward, we have:

\[ W = F \cdot \Delta r = mg \cdot h \]

\[ = mgh \]
• Now imagine we push the object up a frictionless inclined plane, rather than lifting it straight up:

\[ \text{In this case, we need to apply a force of } mg\sin\theta \text{ to move the block} \]
  – Note that this is less than the force needed when lifting straight up – it’s easier to move things up ramps!

• But we need to apply the force for a distance of \( d = \frac{h}{\sin\theta} \), so the work we have to do is \( mgh \) – the same as for a straight lift!
• Let’s try again, but this time apply the force horizontally:

\[ W = F \cdot \Delta r = F \cdot d\mathbf{i} \]

\[ = \left( F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \right) \cdot d\mathbf{i} = F_x d = F_x \frac{h}{\sin \theta} \]

• Balancing forces in the \( x \) direction requires \( F_x \) to be \( mgsin\theta \), so \( W = mgh \) yet again
• So no matter what we do, the work required to lift the object is always $mgh$
  – The same result holds if we use a system of pulleys, or any other mechanical device to raise the mass

• This makes sense when one thinks about dropping the object from a height $h$
  – The object will hit the ground with a velocity that depends only on $h$, not on exactly how it got to that height in the first place
  – In other words, we can think of the work we did on the object as giving it the potential to strike the ground at a given velocity
Work Done By a Variable Force

• So far, we’ve considered the work done by a constant force
• A common example where the force depends on position is a spring
• Many (but not all!) springs obey Hooke’s Law, which tells us that:
  – A spring has a natural, unstretched length. When the spring is at this length, it exerts no force
  – When the length is change by a distance \( x \), the spring exerts a force of magnitude:
    \[
    F = -kx \tag{“Spring constant”}
    \]

• Note that this force is always trying to push the spring back toward its natural length
• How much work must we supply to stretch a spring a distance \( d \)?

• To find out, consider that to move the spring a small distance (a distance small enough that the force is nearly constant throughout) requires a small amount of work:

\[
\frac{dW}{dx} = F(x)
\]

• We can then find the total work by integration:

\[
W = \int_{x_i}^{x_f} F(x) \, dx
\]
• We take the end of the spring to lie originally at \( x = 0 \), and

\[
W = \int_{0}^{d} F(x) \, dx = \int_{0}^{d} kx \, dx
\]

Note the sign of the force – we are applying a force that *opposes* the force of the spring

• Completing the integration, we find:

\[
W = \int_{0}^{d} kx \, dx = \frac{1}{2} kd^2
\]

• If instead we compress the spring, the work is:

\[
W = \int_{0}^{-d} kx \, dx = \frac{1}{2} kd^2
\]