Lecture 19: More on Work and Energy

• So far, we’ve considered the work done by a constant force
• A common example where the force depends on position is a spring
• Many (but not all!) springs obey Hooke’s Law, which tells us that:
  – A spring has a natural, unstretched length. When the spring is at this length, it exerts no force
  – When the length is change by a distance $x$, the spring exerts a force of magnitude:
    \[ F = -kx \]
    
    “Spring constant”

• Note that this force is always trying to push the spring back toward its natural length
• How much work must we supply to stretch a spring a distance $d$?

• To find out, consider that to move the spring a small distance (a distance small enough that the force is nearly constant throughout) requires a small amount of work:

$$dW = F \cdot dr = F(x) \, dx$$

• We can then find the total work by integration:

$$W = \int_{x_i}^{x_f} F(x) \, dx$$
• We take the end of the spring to lie originally at \( x = 0 \), and

\[
W = \int_{0}^{d} F(x) \, dx = \int_{0}^{d} kx \, dx
\]

Note the sign of the force – we are applying a force that opposes the force of the spring.

• Completing the integration, we find:

\[
W = \int_{0}^{d} kx \, dx = \frac{1}{2} kd^2
\]

• If instead we compress the spring, the work is:

\[
W = \int_{0}^{-d} kx \, dx = \frac{1}{2} kd^2
\]
Power

• For both the case of lifting something against gravity, or stretching a spring, we’ve seen that the work doesn’t depend on *how* the object is moved, but rather only on the difference between the initial and final positions.

• The work also doesn’t depend on how *quickly* the object is moved.

• Intuitively, though, we know there’s a difference between lifting something quickly and lifting it slowly.

• This is expressed by the concept of *power*, the rate at which work is done:

\[ P = \frac{W}{t} \]
• Starting from this definition, we can find the power supplied by a given force:

\[ P = \frac{W}{t} = \frac{F \cdot \Delta r}{t} = F \cdot \frac{\Delta r}{t} = F \cdot v \]

• Power has units of Newtons x meters per second, or Joules per second
• One Watt is defined as one Joule per second
  – English units include horsepower, defined as

\[ 1\text{hp} = 550 \frac{\text{ft} \cdot \text{lb}}{\text{s}} = 746\text{W} \]
Example

- The elevator in a 100-story building is required to lift a load of 1000kg from the ground to the top floor in 60s. How powerful must its motor be?
- The total work done is

\[ W = mgh \]

- At about 3.5m/story, this comes to:

\[ W = 1000\text{kg} \cdot 9.8\text{m/s}^2 \cdot 350\text{m} = 3.4\text{MJ} \]

- The power is then:

\[ P = \frac{W}{t} = \frac{3.4\text{MJ}}{60\text{s}} = 57\text{kW} = 77\text{hp} \]
Net Work

• Last time, we explored the work done by a single force
• Now we want to consider the total work done on an object
• For all the examples we’ve done do far, this total is 0
  – For example, we discussed lifting an object to a certain height, and showed that the work required was mgh
  – But gravity was also doing work on the object during that process, with magnitude –mgh, so the total work done on the object was zero
• For all these zero-work examples, the speed of the object didn’t change
• So what happens when the total work is non-zero?
The Work-Energy Theorem

• In general, the total work done in moving an object from \( r_1 \) to \( r_2 \) is:

\[
W_{\text{net}} = \int_{r_1}^{r_2} \mathbf{F}_{\text{net}} \cdot d\mathbf{r} = \int_{x_1}^{x_2} F_{\text{net},x} \, dx + \int_{y_1}^{y_2} F_{\text{net},y} \, dy + \int_{z_1}^{z_2} F_{\text{net},z} \, dz
\]

• Let’s start by considering just the \( x \) coordinate:

\[
\int_{x_1}^{x_2} F_{\text{net},x} \, dx = \int_{x_1}^{x_2} ma_x \, dx = \int_{x_1}^{x_2} m \frac{dv_x}{dt} \, dx
\]

• Using the chain rule, we have:

\[
\int_{x_1}^{x_2} m \frac{dv_x}{dt} \, dx = \int_{x_1}^{x_2} m \frac{dx}{dt} \frac{dv_x}{dx} \, dx = \int_{x_1}^{x_2} m v_x \frac{dv_x}{dx} \, dx
\]
• We can now change our variable of integration from \( x \) to \( v \):

\[
\int_{x_1}^{x_2} F_{\text{net},x} \, dx = \int_{v_{x,1}}^{v_{x,2}} m v_x \, dv_x = \frac{1}{2} m \left( v_{x,2}^2 - v_{x,1}^2 \right)
\]

• Going through similar steps for the \( y \) and \( z \) variables, we find:

\[
W_{\text{net}} = \frac{1}{2} m \left( v_{x,2}^2 - v_{x,1}^2 \right) + \frac{1}{2} m \left( v_{y,2}^2 - v_{y,1}^2 \right) + \frac{1}{2} m \left( v_{z,2}^2 - v_{z,1}^2 \right)
\]

\[
= \frac{1}{2} m \left[ \left( v_{x,2}^2 + v_{y,2}^2 + v_{z,2}^2 \right) - \left( v_{x,1}^2 + v_{y,1}^2 + v_{z,1}^2 \right) \right]
\]

\[
= \frac{1}{2} m \left( v_2^2 - v_1^2 \right)
\]
• The quantity $\frac{1}{2}mv^2$ is defined as the \textit{kinetic energy} of an object

• Thus we have proved the \textit{work-energy} theorem:

\begin{center}
\textbf{The net work done on an object is equal to the change in kinetic energy of the object}
\end{center}

\begin{center}
$(W = \Delta KE)$
\end{center}

• A few notes about kinetic energy:
  – It is a scalar, not a vector (unlike momentum)
  – It depends only on the speed of an object, not the direction of the velocity vector
  – It can never be negative (although the \textit{change} in kinetic energy can be either negative or positive)
Example 1

- A 1000-kg sports car claims to be able to go from 0 to 60 mph in 5.0s. How powerful must its engine be?

- We use the work-energy theorem to tell us that:

\[ W = \Delta KE = \frac{1}{2} m (v_f^2 - v_i^2) \]

\[ = \frac{1}{2} \cdot 1000\text{kg} \cdot \left( \frac{60\text{mi}}{\text{hr}} \cdot \frac{1609\text{m}}{1\text{mi}} \cdot \frac{1\text{hr}}{3600\text{s}} \right)^2 = 3.6 \times 10^5 \text{J} \]

- The power then is just:

\[ P = \frac{W}{t} = \frac{3.6 \times 10^5 \text{J}}{5.0\text{s}} = 72\text{kW} = 97\text{hp} \]

- Question: why, then, do some sports cars have several-hundred hp engines?
• There are two parts to the answer:

1. We didn’t consider any resistive forces in deriving the result. Friction with the ground, and (especially at higher speeds) air resistance need to be overcome

2. What we derived was the *average* power over the interval, not the instantaneous power at the end

• The car’s acceleration was:

\[
a_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{27 \text{ m/s}}{5 \text{ s}} = 5.4 \text{ m/s}^2
\]

• Thus the force on the car was \( F = ma = 5400 \text{ N} \)

• Near the final speed, the instantaneous power is:

\[
P = Fv = 5400 \text{ N} \cdot 27 \text{ m/s} = 1.46 \times 10^5 \text{ W} = 195 \text{ hp}
\]