Lecture 24: Energy and Equilibrium

• We might better understand the consequences of mechanical energy conservation by looking at the situation graphically.

• Consider some random potential $U(x)$, and several possible total energies:

$$U(x)$$

$\begin{align*}
E_1 & \quad x_0 x_1 \\
E_2 & \quad x_2 x_3 x_4 x_5 x_6 \\
E_3 & \quad x_7 x_8 \\
\end{align*}$$
• Let’s consider what happens to a particle with each total energy:

1. A particle with energy $E_1$ might be found anywhere on the $x$-axis. It’s speed will be greatest at $x_4$, and least below $x_0$.

2. A particle with energy $E_2$ will never be found at $x < x_1$. $x_1$ is called a “turning point” for this particle.

3. A particle with energy $E_3$ might be found in the region between $x_2$ and $x_6$, or between $x_7$ and $x_8$, but nowhere else.

4. A particle with energy $E_4$ can only exist in the region between $x_3$ and $x_5$.

5. A particle with energy $E_5$ can only exist at rest at point $x_4$. 
Analogies: Roller coaster

- An intuitive understanding can be gained by considering a ride on a roller coaster.
- First, a mechanical device pulls your car up a hill.
  - This device does work on your car, and hence adds energy.
- At the top of the hill you are moving slowly.
  - Energy is nearly all potential.
- As the car goes down the hill, its speed increases.
  - Potential energy is being “traded” for potential energy.
- Car is going fastest at bottom of the hill.
- If energy is lost (due to friction, for example) car will never be able to make it back to top of first hill.
  - Since real roller coasters always have some friction, first hill is the highest.
Equilibrium

• Looking again at our plot of potential energy versus position:

• We can find the force acting on the particle by differentiating:
• Note that there are several positions at which the force is zero. If an object is placed at any of these positions (with zero velocity) it will never move unless a different force (i.e., not the force associated with the potential energy) acts on it
  – The object is said to be in equilibrium

• This is true whenever \( \frac{dU}{dx} = 0 \)
  – i.e, whenever the potential energy is at a local maximum or minimum

• But all equilibria are not equal!
Types of Equilibria

- The key difference between types of equilibria is what happens when an object is displaced slightly from the equilibrium position.

- Consider an object near a minimum of the potential:

- If the object moves to the left of the minimum, it feels a force to the right.
- If it moves to the right, it feels a force to the left.
- i.e., the object is drawn back to the equilibrium position.
- This is called a *stable* equilibrium.
• If the object is near a maximum of the potential energy, the situation is reversed:

- If the object moves to the left of the minimum, it feels a force to the left
- If it moves to the right, it feels a force to the right
  - i.e., the object is drawn away from the equilibrium position
- This is called a \textit{unstable} equilibrium
• The type of equilibrium we’re in can be determined by evaluating \( \frac{d^2 U}{dx^2} \)
  
  – If it’s negative, the equilibrium is unstable
  
  – Positive means the equilibrium is stable
  
  – If it’s zero, the equilibrium is neutral. The object will feel no force at all near the equilibrium point
    
    • This really means there’s an entire set of equilibrium points that form an “equilibrium region”