Lecture 28: Introduction to Rotational Dynamics

• We previously derived scalar relationships between linear and rotational kinematic quantities
  – e.g., \( v = \omega r \)

• The vector version can be derived starting from the following diagram:
• At this moment (remember that the axis of rotation, and therefore \( \omega \), might change direction later), the magnitude of the linear velocity is:

\[
v = \omega r = \omega R \sin \theta
\]

• We also note that \( v \) is perpendicular to both \( R \) and \( \omega \)
  – The combination of these two facts allows us to conclude that \( v \) is the cross product of \( \omega \) and \( R \)

• Using the right-hand rule to get the correct direction, we find:

\[
v = \omega \times R
\]

• From this, we can also find that:

\[
a = \frac{d(\omega \times R)}{dt} = \frac{d\omega}{dt} \times R + \omega \times \frac{dR}{dt}
\]

\[
= a \times R + \omega \times v = a \times R + \omega \times \omega \times R
\]
Rotational Dynamics

- Now that we know how to describe rotational motion, we can explore the causes of this motion.
- Let’s start with a simple case -- a single particle connected to a light rod of length $r$ with one end fixed at the origin, to which an external force is applied:
• A force diagram for the particle would look like this:

![](image)

• Since we’re interested in the angular acceleration of this particle about the origin, we don’t care about the net force in the $r$ direction.

• The acceleration in the $\phi$ direction is given by:

$$a_\phi = \frac{F \sin \theta}{m}$$
• But we know the relationship between the tangential acceleration and $\alpha$:

$$a_\phi = \alpha r$$

$$\frac{F \sin \theta}{m} = \alpha r$$

• Though it appears arbitrary right now, I’ll multiply both sides by $r$:

$$rF \sin \theta = mr^2 \alpha$$

This quantity is called *torque* ($\tau$), and is analogous to force for linear motion.

This is called the *moment of inertia* ($I$), and is analogous to mass.
One can better see why torque and moment of inertia are useful quantities by considering a two-particle system:

with the following free-body diagrams:
• The net force in the tangential direction for each particle is then:

\[ m_1 : F \sin \theta - T_{12} \sin \xi_1 \]
\[ m_2 : T_{12} \sin \xi_2 \]

• So Newton’s Second Law tells us that:

\[ a_1 = \frac{F \sin \theta - T_{12} \sin \xi_1}{m_1} \]
\[ a_2 = \frac{T_{12} \sin \xi_2}{m_2} \]

• Would help to find a relationship between \( \sin \xi_1 \) and \( \sin \xi_2 \)
• Looking again at our two-body system:

• It’s clear that $r_{1\perp} = r_{2\perp}$, so:

$$r_1 \sin \xi_1 = r_2 \sin \xi_2$$

$$T_{12} r_1 \sin \xi_1 = T_{12} r_2 \sin \xi_2 = c$$

• Plugging this definition of $c$ into our expressions for tangential acceleration, we have:

$$a_1 = \frac{F \sin \theta - c / r_1}{m_1}$$

$$a_2 = \frac{c / r_2}{m_2}$$
Since our two-object system is a rigid one, we know that both masses follow the same rotational kinematics.

In particular, $\alpha_1 = \alpha_2$

\[
\alpha = \frac{a_1}{r_1} = \frac{F \sin \theta - c / r_1}{r_1 m_1} = \frac{a_2}{r_2} = \frac{c}{r_2^2 m_2}
\]

\[
c \left( \frac{1}{r_2^2 m_2} + \frac{1}{r_1^2 m_1} \right) = \frac{F \sin \theta}{r_1 m_1}
\]

\[
c = \frac{F \sin \theta}{r_1 m_1} \left( \frac{m_1 m_2 r_1^2 r_2^2}{r_1^2 m_1 + r_2^2 m_2} \right) = \frac{F \sin \theta m_2 r_1 r_2^2}{r_1^2 m_1 + r_2^2 m_2}
\]

\[
\alpha = \frac{c}{r_2^2 m_2} = \frac{r_1 F \sin \theta}{r_1^2 m_1 + r_2^2 m_2}
\]

\[
r_1 F \sin \theta = \left( r_1^2 m_1 + r_2^2 m_2 \right) \alpha
\]

\[
\tau = I \alpha
\]
**Notes on Torque**

- We’ve seen that if we define torque as \( rF\sin \theta \), and the moment of inertia as \( \sum_{i=1}^{N} m_i r_i^2 \), we end up with an equation that looks just like Newton’s Second Law.

- There is a crucial difference, though:
  - Mass is an intrinsic property of an object, or system of objects
  - Moment of inertia, on the other hand, also depends on the axis about which we’re rotating

- In other words, I really should have written the equation as:

\[
\tau_z = I_z \alpha_z
\]

- to emphasize that I was considering rotations about the \( z \) axis
Torque as a Vector

• The fact that $I$ depends on which axis we’re talking about means that one can’t write an equation like this:

$$\tau = I\alpha$$

• But one can still define a vector form of torque:

$$\tau = \mathbf{r} \times \mathbf{F}$$

this is consistent with the magnitude of the torque we’ve defined

• One final note: unless mass and force, moment of inertia and torque depend on the frame of reference