Lecture 38: Gravitational Potential Energy

- The force of gravity between two objects depends only on the masses of the objects and the distance between them.
- If we take one of the objects to be fixed, the work done by gravity in moving the other object from distance \( r_1 \) to \( r_2 \) is:

\[
W_{12} = \int_{r_1}^{r_2} F(r)dr
\]

and the work done to move it back to \( r_2 \) is:

\[
W_{21} = \int_{r_2}^{r_1} F(r)dr = -\int_{r_1}^{r_2} F(r)dr = -W_{12}
\]

- So we see that the work done for a round trip from \( r_1 \) to \( r_2 \) and back is 0.

Gravity is a conservative force, so we can define a potential energy associated with it.
• We can find the change in potential energy between $r_1$ and $r_2$ using the definition:

$$\Delta U = U(r_2) - U(r_1) = -W_{12} = -\int_{r_1}^{r_2} \frac{GMm}{r^2} dr$$

$$= GMm \left[ \frac{1}{r} \right]_{r_1}^{r_2} = GMm \left[ \frac{1}{r_2} - \frac{1}{r_1} \right]$$

• As always, only differences in potential energy are physically important, and we are free to set $U$ to 0 at any $r$ we find convenient

• The most common choice is to set $r = \infty$. With that choice the potential energy at any other distance is:

$$U(r) = \frac{-GMm}{r}$$
Escape Speed

• The gravitational potential energy due to a single object looks like:

\[ V(r) = \begin{cases} \frac{-20}{r} & \text{for } -20 < r < 0 \\ \frac{-15}{r} & \text{for } -15 < r < 0 \\ \frac{-10}{r} & \text{for } -10 < r < 0 \\ \frac{-5}{r} & \text{for } -5 < r < 0 \\ \frac{0}{r} & \text{for } 0 < r < 0 \\ \frac{5}{r} & \text{for } 5 < r < 0 \\ \frac{10}{r} & \text{for } 10 < r < 0 \\ \frac{15}{r} & \text{for } 15 < r < 0 \\ \frac{20}{r} & \text{for } 20 < r < 0 \\ \end{cases} \]

• If a particle has total energy \( E > 0 \), the kinetic energy is positive for all values of \( r \)
  
  – Such a particle can “escape” from the gravitational attraction of the object

• If a particle begins a distance \( R \) from the object, the initial speed needed for escape is:

\[ E = \frac{1}{2} mv^2 - \frac{GMm}{R} = 0; \quad v = \sqrt{\frac{2GM}{R}} \]
Example: Pioneer space probe

- The Pioneer space probe was the first man-made object to leave the solar system
  - So it needed to escape from both the Earth and the sun. Which requires the greatest initial velocity, if we start from the Earth’s surface?

- Earth:
  
  \[ v = \sqrt{\frac{2GM_E}{r_E}} = \sqrt{\frac{2 \cdot 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \cdot 5.98 \times 10^{24} \text{kg}}{6.37 \times 10^6 \text{m}}} \]
  
  \[ = 11.2 \text{km/s} \]

- Sun
  
  \[ v = \sqrt{\frac{2 \cdot 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \cdot 1.99 \times 10^{30} \text{kg}}{1.50 \times 10^{11} \text{m}}} \]
  
  \[ = 42.1 \text{km/s} \]
Gravitational Potential Energy of Many-body Systems

- We can extend the two-body gravitational potential energy to an arbitrary number of particles by considering the work required to assemble the system “piece-by-piece”
- Example: A three particle system
- We want to find the gravitational potential energy of the following system:
• We begin with all three masses infinitely far from each other
• Then we move $m_1$ to its final position
  – We do no work here, since there is no gravitational force
• Next we bring in $m_2$ at constant velocity, while $m_1$ remains stationary
  – The work done is
  \[ W = -\frac{Gm_1m_2}{r_{12}} \]
  – Negative since we are preventing $m_2$ from accelerating toward $m_1$
• Finally we bring in $m_3$
• Now we have to fight against gravity from both $m_1$ and $m_2$, so the work done is:
  \[ W = -\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} \]
• This means that the total gravitational potential energy of the system is:

\[ U = -\frac{Gm_1 m_2}{r_{12}} - \frac{Gm_1 m_3}{r_{13}} - \frac{Gm_2 m_3}{r_{23}} \]

• In other words, it would take that much energy to move the masses infinitely far from each other
  
  – Note that the result does not depend on which order the masses are brought in from infinity, nor on the path taken while doing so

• This type of potential energy is known as “binding energy”, and is a very useful concept in atomic and nuclear physics as well