Variation of Pressure with Depth (For a Liquid)

- Consider a liquid at rest near the Earth’s surface:

Consider the forces acting on this tiny element of liquid $dm$

- Since the liquid remains at rest, the net force must be zero:

$$F_x = A_{\text{side}} (p_1 - p_2) = 0$$
$$F_y = A_{\text{top}} p_4 - dmg - A_{\text{bot}} p_3 = 0$$

- So the pressure changes only with depth (not horizontal position)
To determine the variation of $p$ with depth ($y$) we assign $dm$ a thickness $dy$, and note that $A_{\text{top}} = A_{\text{bot}}$ (we’ll just call it $A$ from now on)

- So that $dm = \rho A dy$

Our force-balance equation is then:

\[
Ap(y) - dmg - Ap(y + dy) = 0
\]

\[
A[p(y) - p(y + dy)] = \rho g A dy
\]

\[
-dp = \rho g dy
\]

\[
p(y) = \int -\rho g dy = -\rho gy + C = -\rho gy + p_o
\]

So the pressure increases linearly with depth in the liquid

- Note that we assumed $\rho$ was constant, which means we approximated the fluid as completely incompressible
Variation of Pressure with Depth (For a Gas)

- If the fluid is a gas, we can no longer assume that the density is the same everywhere.
- To proceed, we make the assumption that the temperature of the gas is constant.
  - Not really true for the Earth’s atmosphere…
- Then the ideal gas law says:
  \[ pV = nRT \]
  These are all constants
- So,
  \[
  p \frac{m}{\rho} = C \\
  p = C' \rho \\
  \rho(y) = \rho_o \frac{p}{p_o}
  \]
The force-balance equation in the $y$ direction is the same as that for a liquid, leading to:

$$dp = -\rho(y)g\,dy = -\rho_o \frac{p}{p_o}g\,dy$$

The quantity $a \equiv g \frac{p_o}{\rho_o}$ determines how quickly $p$ decreases with increasing $h$.

For Earth’s atmosphere, $a = 8.55\text{km}$
Pascal’s Principle

- Consider a flexible container filled with fluid:
- The pressure varies throughout the fluid due to gravity
- Now, we apply an additional force to any part of the container:
- At the area where the force is applied, the pressure of the fluid must increase by $\Delta p$ (to keep the container wall in equilibrium)

Pascal’s principal states that when this happens, the pressure everywhere in the fluid increases by $\Delta p$
Taking Advantage of Pascal’s Principal

- Let’s say our container looks like this:

\[ \text{This arm has cross-sectional area } A_S \]

- We apply a force \( F_1 \) to the left piston
  - This in turn applies an additional pressure \( \Delta p = \frac{F_1}{A_S} \) to the fluid
- The upward force on the right-hand piston increases by:

\[ F_2 = \Delta p A_L = F_1 \frac{A_L}{A_S} \]
• We see that the liquid acts as a “force multiplier”
  – A small external force applied to one part of the fluid results in a large force being applied somewhere else
  – This is how hydraulic systems work
Archimedes’ Principle

• Let’s consider now the forces acting throughout a static fluid

• Look at a small element $dV$ of the fluid (with mass $dm$):

  • $F_B$ is the upwards “buoyant” force exerted by the rest of the fluid on $dm$
    - We know this force must exist, since $dm$ is not accelerating downward
    - We also know that the magnitude of the buoyant force is equal to the weight of the fluid contained in $dm$
• What happens if we replace the fluid in $dV$ with some other material (with different density, but same size and shape)?

  i.e., the new object “displaces” the fluid in $dV$

• The rest of the fluid isn’t “aware” of this change
  – And thus the buoyant force supplied by the fluid doesn’t change

• This leads us to Archimedes’ Principle:

The buoyant force exerted on an object is equal to the weight of the fluid displaced by that object
Floating a Boat

- Archimedes’ principle explains how a boat made of steel can float:
  - The mass of the boat is equal to the mass of the displaced water
    - If the boat is loaded so full that the average density of the material in its volume becomes greater than the density of water, it will sink!
  - Sometimes we also consider the torque produced by the buoyant force
    - For this, we can treat the force as though it acts at the “center of buoyancy”, which is the center of mass of the displaced fluid