Physics 321  
Solutions for Midterm #3

1) Two particles of mass $M$ and $m$ exert a force on each other. This force acts on the line between the particles, and has a magnitude $F = -k/r^3$, where $r$ is the distance between the particles. $M$ is much greater than $m$, so that it can be treated as stationary.

a) Find the potential $U(r)$ and effective potential $V(r)$ associated with this force

We know that the force is given by $F = -\nabla U$, but for central forces this simplifies to:

$$F = -\frac{\partial U}{\partial r} e_r = \frac{-k}{r^4} e_r$$

$$U = \int \frac{k}{r^4} dr = \frac{-k}{3r^3} + C$$

Making the usual choice that $U$ is zero as $r$ goes to infinity, we set $C$ to 0. Thus:

$$U = \frac{-k}{3r^3}$$

$$V = U + \frac{l^2}{2mr^2} = \frac{-k}{3r^3} + \frac{l^2}{2mr^2}$$

b) Determine the radius of a circular orbit for a given angular momentum $l$.

A circular orbit must be at an equilibrium point for radial motion, meaning that:

$$\frac{dV}{dr} = 0$$

$$\frac{dV}{dr} = \frac{k}{r^4} - \frac{l^2}{mr^3} = 0$$

$$\frac{k}{r} = \frac{l^2}{mr}$$

$$r = \frac{mk}{l^2}$$
c) Determine the energy of the circular orbit found in part (b).

Since there is no motion in the radial direction for a circular orbit, the total energy is just equal to the effective potential:

\[
E = V(r_{\text{orb}}) = \frac{-k}{3 \left( \frac{mk}{l^2} \right)^3} + \frac{l^2}{2m \left( \frac{mk}{l^2} \right)}
\]

\[
= -\frac{l^6}{3m^3k^2} + \frac{l^6}{2m^3k^2} = \frac{1}{6} \left( \frac{3l^6}{m^3k^2} - \frac{2l^6}{m^3k^2} \right) = \frac{1}{6} \frac{l^6}{m^3k^2}
\]

d) Is the circular orbit stable or unstable?

We can determine the stability of the orbit by calculating \( \frac{d^2V}{dr^2} \) at the orbital radius:

\[
\frac{d^2V}{dr^2} = \frac{d}{dr} \left( \frac{k}{r^4} - \frac{l^2}{mr^3} \right) = -\frac{4k}{r^5} + \frac{3l^2}{mr^4} = \frac{1}{r^4} \left( \frac{3l^2}{m} - \frac{4k}{r} \right)
\]

Since \( r \) is always positive, we can determine the sign of the second derivative from the term in parentheses:

\[
\left( \frac{3l^2}{m} - \frac{4k}{r} \right) = \left( \frac{3l^2}{m} - \frac{4k}{mk \frac{l^2}{l^2}} \right) = \left( \frac{3l^2}{m} - \frac{4l^2}{m} \right) = -\frac{l^2}{m}
\]

This is less than 0, so the orbit must be unstable.
2) A mass $m$ is attached to a spring with force constant $k$ and relaxed length $l$. The spring is suspended from a support that is moving upwards with a constant acceleration $a$. Assume that all motion is in the vertical direction.

![Diagram of a mass attached to a spring suspended from a support moving upwards with a constant acceleration.]

a) Find the Hamiltonian for this system

We take the extension of the spring from its natural length, $s$, to be the generalized coordinate. The Cartesian coordinate $y$ is then given by:

$$y = \frac{1}{2}at^2 - l + s = s - l + \frac{1}{2}at^2$$

so the kinetic and potential energy is:

$$T = \frac{1}{2}my^2 = \frac{1}{2}m(\dot{s} + at)^2$$

$$U = mgy + \frac{1}{2}ks^2 = mg\left(s - l + \frac{1}{2}at^2\right) + \frac{1}{2}ks^2$$

And therefore the Lagrangian is:
\[ L = T - U = \frac{1}{2} m y^2 = \frac{1}{2} m (s + at)^2 - mg \left( s - l + \frac{1}{2} at^2 \right) - \frac{1}{2} ks^2 \]

\[ p_s = \frac{\partial L}{\partial \dot{s}} = m (s + at) \]

\[ \dot{s} = \frac{p_s}{m} - at \]

\[ H = p_s \dot{s} - L = p_s \left( \frac{p_s}{m} - at \right) - \frac{1}{2} \frac{p_s^2}{m} + mg \left( s - l + \frac{1}{2} at^2 \right) + \frac{1}{2} ks^2 \]

\[ = \frac{1}{2} \frac{p_s^2}{m} - p_s at + mg \left( s - l + \frac{1}{2} at^2 \right) + \frac{1}{2} ks^2 \]

b) Determine Hamilton’s Equations of motion

The equations of motion are:

\[ \dot{s} = \frac{\partial H}{\partial p_s} = \frac{p_s}{m} - at \]

\[ \dot{p}_s = -\frac{\partial H}{\partial s} = -mg - ks \]

c) Is energy conserved? Explain why or why not.

Since the Hamiltonian depends explicitly on time, energy is not conserved. (Intuitively, something external to the system must be supplying energy to accelerate the spring’s support upwards).