Lecture 12: More on the Gravitational Potential

- Using the concept of gravitational potential, we can determine how much initial velocity an object must have if it is to escape the gravitational field of another object
  - “Escape” means it’s able to move infinitely far from the object generating the field
- For example, the moon is “infinitely” far from the Earth, so the Apollo spacecraft needed to escape from the Earth’s gravitational field to get there
- Initial state: near the surface of the Earth, with some velocity $v$
- Final state: infinitely far from the Earth, with (at least) zero velocity

\[ E_i = \frac{1}{2}mv^2 - \frac{GMm}{R_E} = E_f \geq 0 - \frac{GMm}{\infty} = 0 \]

\[ v \geq \sqrt{\frac{2GM}{R_E}} \]
Graphical Representation of Force and Potential

- We can gain an intuitive picture of a force field and potential as follows:
  1. At a random point in space, measure the force vector, which can be represented by an arrow.
  2. Move a small distance in the direction of the force, and repeat the measurement.
  3. Repeat steps 1 and 2 many times.
  4. Draw a line connecting all the arrows.

- The line you’ve drawn is called a “line of force”.
- By repeating the while process starting from different points, one creates a picture of the force.
- Note that lines of force can never cross one another.
• Once the lines of force are drawn, trace out a path that is perpendicular to all lines of force.
• Recall that the work done by the force is equal to the change in potential energy:

\[ W = \Delta U \]

• But on the path we’ve drawn, the direction of motion is always perpendicular to the force, so:

\[ W = \int \mathbf{F} \cdot d\mathbf{r} = 0 \]

• This means that \( \Delta U = 0 \), so that \( U \) is constant everywhere on the path.
• Such a path is called an *equipotential surface*. 
Gauss’ Law For Gravitation

• We can define the gravitational field flux through a surface as:

\[ \Phi_m = \int_S \mathbf{g} \cdot \mathbf{n} \, da \]

where \( \mathbf{n} \) is a unit vector perpendicular to the surface at each point.

• For the case of a closed surface around a point mass \( m \), we have:

\[ \Phi_m = \int_S \mathbf{g} \cdot \mathbf{n} \, da = \int_S \left( \frac{-Gm}{r^2} \right) \mathbf{e}_r \cdot \mathbf{n} \, da \]

\[ = -Gm \int_S \frac{\cos \theta}{r^2} \, da \]
• If we take the special case where the surface is a sphere, the integral is easy:

\[
\Phi_m = -Gm \int \frac{1}{r^2} da = \frac{-Gm}{r^2} \int da
\]

\[= -4\pi Gm\]

• Physically, we can interpret the flux as the “number” of gravitational field lines passing through the surface
  - But since the lines start at a point, and extend an infinite distance away, the flux can’t depend on the shape of the surface that encloses our point

• Thus we have the general result, for any closed surface around a point mass,

\[\Phi_m = -4\pi Gm\]
• This can be easily extended to the case where there are $N$ point masses inside the surface. Since the field is linear,

$$\Phi_m = -4\pi G \sum_{i=1}^{N} m_i$$

• For a continuous distribution of matter, it’s:

$$\Phi_m = -4\pi G \int_{V} \rho (\mathbf{r}) dV$$

• For problems involving symmetric distributions of matter, Gauss’ Law is a useful shortcut to finding the field
Example: Field Due to a Sphere

• Assume we have a spherically symmetric mass distribution (with the density varying as a function of the distance from the sphere’s center)

• We want to find the field at any point external to the sphere
  – The symmetry of the problem makes it a good candidate for Gauss’ Law
  – We should choose a surface that reflects this symmetry:

\[ \int_S \mathbf{g} \cdot \mathbf{n} \, da = -4\pi G M \]

From symmetry, we know that:
\[ \mathbf{g} = g_r(r) \mathbf{e}_r \] and \[ \mathbf{n} = \mathbf{e}_r \]
• Therefore, the integral becomes much simpler:

\[ \int g \cdot n \, da = \int g(r) \, da = 4\pi r^2 g(r) \]

\[ 4\pi r^2 g(r) = -4\pi GM \]

\[ g(r) = \frac{-GM}{r^2} \]

• From this, we see that the field due to a sphere is exactly the same as if all the mass were concentrated at the center of the sphere
  – This is not true for other shapes

• This result was very important for Newton, since it justified his treatment of the Earth as a point mass when calculating the motion of the moon
Poisson’s Equation

• We can look at Gauss’ Law another way to find another important property of the gravitational potential:

\[ \int_{s} g \cdot n \, da = \int_{v} \nabla \cdot g \, dv \]

This is just the divergence theorem

\[ = \int_{v} (\nabla \cdot (-\nabla \Phi)) \, dv = -\int_{v} \nabla^{2} \Phi \, dv \]

\[ = -4\pi G \int_{v} \rho \left( r' \right) \, dv \]

• For the last relation to hold for an arbitrary volume \( V \), the integrands must be the same everywhere:

\[ \nabla^{2} \Phi = 4\pi G \rho \left( r' \right) \]
Laplace’s Equation

• In the special case where there is no material in a region of space, the potential in that region satisfies Laplace’s Equation:

\[ \nabla^2 \Phi = 0 \]

• Intuitively, this is nothing more than the statement that field lines can’t start (or end) in a region where there is no mass

• Mathematically, this gives us a way to determine the potential in any mass-free region
  – As long as the boundary conditions (the value of the potential at the edges of the region) are specified

• In practice, this equation is more useful in calculating electric potentials than for gravitational potentials…