Lecture 14: Calculus of Variations

• The next few lectures may seem like a departure from mechanics (and indeed from physics altogether)
  – But we’ll see the (surprising!) way it all relates very soon
• We’ll start by asking a deceptively simple question: what path represents the shortest distance between two points in a plane?
  – Of course, we know the answer – it’s a line!
  – But, how can we prove it?
• Let’s set up the problem as follows:

The red line is described by some function $y(x)$
• To find the total length of the path from $x_1$ to $x_2$, we first zoom in on a very small segment of the line 
  – So small that any curvature can be neglected

  This element has a length
  \[ ds = \sqrt{(dx)^2 + (dy)^2} \]

• So the total path length is \[ s = \int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} \sqrt{(dx)^2 + (dy)^2} \]

• Multiplying the integrand by \( \frac{dx}{dx} \) gives:

\[
s = \int_{x_1}^{x_2} \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx
\]
• To generalize the problem, we have an integral over some function of \(y'\), and we want to find the function \(y(x)\) that minimizes the integral
  – In other, similar, problems, the integrand may depend directly on \(y\) and \(x\)
• So the general statement of our problem is that we need to find \(y\) such that the following integral is minimized:

\[
J = \int_{x_1}^{x_2} f\{y, y'; x\} \, dx
\]

For the particular example of the shortest distance between two points, \(f\{y, y'; x\} = \sqrt{1 + y'^2}\)

• Assume now that \(y\) is the function we want
  – i.e., it’s the one that minimizes \(J\)
• Then any other function could be written as:

\[
y(\alpha, x) = y(x) + \alpha \eta(x)
\]
• Note that the function $\eta$ is not completely arbitrary, since we’re only considering functions that go through $P_1$ and $P_2$
  – Since $\alpha$ is not restricted to be 0, this means that:

$$\eta(x_1) = \eta(x_2) = 0$$

• Now, since we assumed that $y$ minimizes the integral, what we want is the “functional” $J(\alpha)$ to be minimized when $\alpha = 0$:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\} dx$$

$$\left. \frac{dJ}{d\alpha} \right|_{\alpha=0} = 0$$

$$\frac{d}{d\alpha} \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\} dx = 0$$
• The derivative is made simpler by the fact that the endpoints of the integration are considered to be fixed (constants). So we can move the derivative inside the integral:

\[
\frac{d}{d\alpha} \int_{x_1}^{x_2} f dx = \int_{x_1}^{x_2} \frac{d}{d\alpha} f dx \\
= \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx
\]

• Recall that:
\[
y(\alpha, x) = y(x) + \alpha \eta(x) \\
y'(\alpha, x) = y'(x) + \alpha \eta'(x)
\]
which means that:
\[
\frac{dy}{d\alpha} = \eta; \quad \frac{dy'}{d\alpha} = \eta'
\]
• Therefore,

\[
\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right) dx
\]

• It would be nice if we could factor out the \( \eta \) dependence

• Try integration by parts on the second term in the integral:

\[
\int uv = uv - \int vdu
\]

\[
u = \frac{df}{dy'}; \quad dv = \eta' dx
\]

\[
u = \eta
\]

\[
du = \frac{d}{dx} \frac{df}{dy'} dx; \quad dx = \eta
\]

\[
\int_{x_1}^{x_2} \frac{df}{dy'} \eta' dx = \frac{df}{dy'} \eta \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \frac{df}{dy'} \eta dx
\]

This term vanishes because \( \eta \) is 0 at both \( x_1 \) and \( x_2 \).
So we’re left with:

\[
\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \eta - \frac{d}{dx} \frac{\partial f}{\partial y'} \eta \right) dx
\]

\[
= \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta dx
\]

Now, for the function that minimizes \( J \), we want the above expression to be zero when \( \alpha = 0 \)

- We get the desired result if:

\[
\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0
\]

This is known as Euler’s equation for determining the function \( y \)
Does it Really Work?

• Let’s return now to the problem we started with – the shortest path between two points in a plane.

• In that case,

\[
f = \sqrt{1 + y''^2}
\]

\[
\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 - \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) = 0
\]

• This implies that:

\[
\frac{y'}{\sqrt{1 + y'^2}} = \text{const} = A
\]

\[
y' = A\sqrt{1 + y'^2}
\]

\[
y'^2 = A^2 \left( 1 + y'^2 \right)
\]

\[
y' = \frac{A}{\sqrt{1 - A^2}} = A'
\]
• We can now integrate to find:

\[ y(x) = A'x + B \]

Note that \( A' \) and \( B \) are determined by the requirements that 
\[ y(x_1) = y_1 \text{ and } y(x_2) = y_2 \]

• It’s a line!
  – …just like we knew all along

• Though the answer in this case was obvious, Euler’s Equation lets us find other minimizing functions that we might not have been able to guess beforehand