Lecture 16: Euler’s Equations for Multi-variable Problems

• Let’s say we’re trying to minimize an integral of the form:

\[ J = \int_{x_1}^{x_2} f \left\{ y_1, y_1', y_2, y_2', \ldots, y_N, y_N'; x \right\} dx \]

• We can start by writing each of the \( y \)'s as we did before:

\[ y_i (\alpha, x) = y_i (0) + \alpha \eta_i (x) \]

Remember that \( y_i(0) \)

\[ \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f \left\{ y_1, y_1', y_2, y_2', \ldots, y_N, y_N'; x \right\} dx \]
• Applying the chain rule, we find:

\[
\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y_1} \frac{\partial y_1}{\partial \alpha} + \frac{\partial f}{\partial y_1} \frac{\partial y_1}{\partial \alpha} \right) dx + \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y_2} \frac{\partial y_2}{\partial \alpha} + \frac{\partial f}{\partial y_2} \frac{\partial y_2}{\partial \alpha} \right) dx
\]

\[+ \ldots + \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y_N} \frac{\partial y_N}{\partial \alpha} + \frac{\partial f}{\partial y_N} \frac{\partial y_N}{\partial \alpha} \right) dx\]

• Which means we have \( N \) integrals of the type encountered in the one-variable problem

• So, for \( \frac{\partial J}{\partial \alpha} \) to be 0, each integral must be zero

• This means we have to satisfy \( N \) Euler equations, of the form:

\[
\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} = 0
\]
Euler’s Equation with Constraints

• In some cases, we want to restrict the set of paths one can take between points \( x_1 \) and \( x_2 \), and find the minimum of \( J \) subject to this constraint
  
  – For example, we might want to find the shortest distance between two points on a surface with a fixed shape

• If this happens, our derivation of the many-variable Euler Equation is altered. We still have:

\[
\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_2} \right) \frac{\partial y_1}{\partial \alpha} \, dx + \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y_2} + \frac{\partial f}{\partial y_3} \right) \frac{\partial y_2}{\partial \alpha} \, dx + \ldots + \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y_N} + \frac{\partial f}{\partial y_N} \right) \frac{\partial y_N}{\partial \alpha} \, dx = 0 \quad \text{when} \quad \alpha = 0
\]

but now the \( y_i \) are not independent, so we can’t assume that each integral is zero individually
• In general, such a constraint can be expressed as a function relating the variables to each other:
  \[ g \{ y_i ; x \} = 0 \]
  
  and there may be an arbitrary number \( m \) of such constraint equations (well, not quite arbitrary – \( m \) must be less than \( N \), the number of variables in the problem)

• For any one of the constraint equations we have:
  \[ g_j \{ y_i ; x \} = 0 \]
  
  \[ dg_j = \left( \sum_{i=1}^{N} \frac{\partial g_j}{\partial y_i} \frac{\partial y_i}{\partial \alpha} \right) d\alpha = \left( \sum_{i=1}^{N} \frac{\partial g_j}{\partial y_i} \eta_i \right) d\alpha = 0 \]

• This last relation holds even if we multiply by an arbitrary function of \( x \):
  \[ \lambda_j (x) \left( \sum_{i=1}^{N} \frac{\partial g_j}{\partial y_i} \eta_i \right) d\alpha = 0 \]
• We can sum over all $m$ constraints, and integrate from $x_1$ to $x_2$, and still get zero:

$$\int_{x_1}^{x_2} \sum_{i,j} \lambda_j(x) \frac{\partial g_j}{\partial y_i} \eta d\alpha = 0$$

• Now we can add this rather exotic-looking form of the number 0 to our equation for $\frac{\partial J}{\partial \alpha}$:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_{i=1}^{N} \left( \left[ \frac{df}{dy_i} + \frac{d}{dx} \frac{df}{dy_i'} + \sum_{j=1}^{m} \lambda_j(x) \frac{\partial g_j}{\partial y_i} \right] \eta_i \right) dx$$

But since the variables are related, we still can’t assume that each term in $[]$ is zero individually.
• However, we can always redefine the variables of the problem such that the first \(N-m\) of them are independent
  
  – The remaining \(m\) variables are related by the \(m\) constraint equations
  
  – For the \(N-m\) independent variables, we must have the terms in [] be zero individually:

\[
\left\{ \frac{\partial f}{\partial y_i} + \frac{d}{dx} \frac{\partial f}{\partial y_i'} + \sum_{j=1}^{m} \lambda_j (x) \frac{\partial g_j}{\partial y_i} \right\} = 0 \text{ for } j \leq N - m
\]

• Since we’ve said nothing so far about the value of the functions \(\lambda_j(x)\), we are free to choose ones that are convenient. In particular, we choose them such that:

\[
\left\{ \frac{\partial f}{\partial y_i} + \frac{d}{dx} \frac{\partial f}{\partial y_i'} + \sum_{j=1}^{m} \lambda_j (x) \frac{\partial g_j}{\partial y_i} \right\} = 0 \text{ for } j > N - m
\]
Combining the expressions on the previous slides, we arrive at Euler’s Equations when constraints are applied:

\[
\left[ \frac{\partial f}{\partial y_i} + \frac{d}{dx} \frac{\partial f}{\partial y'_i} + \sum_{j=1}^{m} \lambda_j (x) \frac{\partial g_j}{\partial y_i} \right] = 0 \text{ for any } j
\]

You may be concerned that there are \(N+m\) variables in the above equations, but only \(N\) equations available

- But remember that the \(m\) equations of constraint are also available

So we can solve for everything – including the \(\lambda\)’s!

Keep in mind that we made a special choice for the \(\lambda\)’s during the derivation. When we apply these ideas to physics we’ll see what they represent.
Example: Geodesic on a Sphere

- Let’s say we want to find the shortest distance between two points, but we have to move on the surface of a sphere between them.
- That means we need to minimize the integral:

\[
J = \int ds = \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2}
\]

\[
= \int \sqrt{1 + y'^2 + z'^2} \ dx \ f
\]

where only paths that satisfy the following constraint are considered:

\[
g(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0
\]
Since $f$ depends on two variables in this problem, we have the following Euler equations:

\[
\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y} = 0
\]

\[
\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial yz} - \lambda \frac{\partial g}{\partial z} = 0
\]

The first equation gives:

\[
0 - \frac{d}{dx} \left[ \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \right] - 2\lambda y = 0
\]

\[
0 - \frac{d}{dx} \left[ \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \right] - 2\lambda z = 0
\]
• Note that we can solve each of these for $\lambda$:

$$\lambda = \frac{1}{2y} \frac{d}{dx} \left[ \sqrt{1 + y'^2 + z'} \right] = \frac{1}{2z} \frac{d}{dx} \left[ \sqrt{1 + y'^2 + z'} \right]$$

$$z \left[ \frac{y''}{\sqrt{1 + y'^2 + z'^2}} - \frac{y'y'' + y'z''}{(1 + y'^2 + z'^2)^{3/2}} \right] =$$

$$y \left[ \frac{z''}{\sqrt{1 + y'^2 + z'^2}} - \frac{z'z'' + y'z''}{(1 + y'^2 + z'^2)^{3/2}} \right]$$

$$z \left[ y''(1 + y'^2 + z'^2) - y'^2 y'' - y'z'z'' \right] =$$

$$y \left[ z''(1 + y'^2 + z'^2) - z'^2 z'' - y'z'z'' \right]$$

$$zy'' + zy''y'^2 + zy''z'^2 - zy''y'' - zy'z'z'' =$$

$$yz'' + yz''y'^2 + yz''z'^2 - yz''z'' - yy'z'z''$$
\[ zy'' + zy''z' - zy'z'z'' = yz'' + yz''y' - yy'z'z'' \]

\[ zy'' + z'y''(zz' + yy') = yz'' + y'z''(zz' + yy') \]

• Note that we haven’t used any information from the equation of constraint yet. We can do so if we take a derivative of \( g \):

\[ \frac{dg}{dx} = 2x + 2yy' + 2zz' = 0 \]

\[ yy' + zz' = 0 \]

• Plugging this back in above gives:

\[ zy'' - xz'y'' = yz'' - xy'z'' \]

• One can verify that a plane passing through the center of the sphere \((Ax + By = z)\) is the solution to the above equation