Lecture 17: Hamilton’s Principle

- Isaac Newton wasn’t the only physicist to think about mechanics
- William R. Hamilton, in 1834 and 1835, came up with his own rule for how things move
- He began by noting that, for a particle moving in a single dimension, the motion can be represented by a path in space-time:
• Motion in more dimensions can be thought of the same way, even though we can’t draw the paths on paper
• In this view, if we can pick the one path (out of the infinite number of possibilities) that the particle takes, we have solved the problem
• Hamilton gave us a rule for doing this, know as Hamilton’s Principle:

  When an object moves through space in a given time interval, it will choose the path that minimizes the integral of the difference between the kinetic and potential energies

• Note that this sounds *nothing* like Newton’s Laws!
• Using the techniques for the calculus of variations that we’ve learned, though, we can test out this crazy idea
The Lagrangian

- The quantity $T-U$ is known as the *Lagrangian*, and written as $L$
- The kinetic energy is a function of velocity, and (usually) the potential energy is a function of position
- So we can typically write:

$$L\{x_i, \dot{x}_i; t\} = T(\dot{x}_i; t) - U(x_i; t)$$

- And according to Hamilton’s Principle, we assert that the following integral must be minimized:

$$J = \int_{t_1}^{t_2} L\{\dot{x}_i, x_i; t\} dt$$
• Applying the Euler equations to this problem, we find that the function $L$ that minimizes the integral is given by:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0$$

Lagrange Equations of Motion

• Note that there is one such equation for each coordinate

• Let’s see if this really works
  – Reconsider projectile motion with the Lagrangian method

\[
U = mgy; \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)
\]

\[
L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy
\]
• Motion in $y$:

$$\frac{\partial L}{\partial y} = -mg; \quad \frac{\partial L}{\partial \dot{y}} = m\ddot{y}$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = -mg - m\ddot{y} = 0$$

$$\ddot{y} = -g$$

• Motion in $x$:

$$\frac{\partial L}{\partial x} = 0; \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 - m\ddot{x} = 0$$

$$\ddot{x} = 0$$

• Exactly the same equations of motions we obtained using Newton’s Laws!
Example 2: Atwood’s Machine

- Consider a pulley with unequal masses strung over it:

\[ U = -m_1 g x_1 - m_2 g (l - x_1) \]

\[ T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 + \frac{1}{2} I \dot{\omega}^2 \]

\[ = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 + \frac{1}{2} I \left( \frac{\dot{x}_1}{R} \right)^2 \]

\[ L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 + \frac{1}{2} I \left( \frac{\dot{x}_1}{R} \right)^2 + m_1 g x_1 + m_2 g (l - x_1) \]

\[ \frac{\partial L}{\partial x_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} = (m_1 - m_2) g - (m_1 + m_2) \ddot{x}_1 - \frac{I}{R^2} \dddot{x}_1 = 0 \]

\[ \dddot{x}_1 = \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{I}{R^2}} \]
Why Lagrangians?

- So it seems that we can get the right answer using the Lagrange Equations of motion
  - But if all it does is get us to the same place as Newton’s Laws, what’s the point?

- There are several advantages, it turns out:
  1. With the Lagrangian technique, we can find the equations of motion for systems where direct application of Newton’s Laws is very difficult
     - Remember the double-pendulum example from the first day of class
     - Also, keep in mind that obtaining the right equation of motion is really the entire battle these days – computers can solve just about any differential equation we throw at them
2. Sometimes, we don’t know the form of the forces acting on an object
   • For example, if a bead is sliding down a curved wire, the force between the wire and bead isn’t known
   • Can’t apply Newton’s Laws in this case!

3. The Lagrangian is a *scalar*, whereas Newton’s Laws give vector equations
   • This means that the Lagrangian is independent of the coordinate system – we’ll get the same results no matter what coordinates we pick
Generalized Coordinates

• In fact, we have tremendous freedom in choosing the coordinates when writing the Lagrangian for a system.

• The general mechanical system will consist of $n$ degrees of freedom:
  - e.g., a projectile has three degrees of freedom, one for each cartesian coordinate.
  - the bob of a pendulum, though, has only one degree of freedom – the angle of the pendulum from the vertical.

• *Any* set of $n$ quantities that completely specifies the state of the system can be taken as “coordinates” for the Lagrangian:
  - They don’t need to be perpendicular to each other, have units of length, or anything else.

• The numbers we choose to describe the system are called *generalized coordinates*. 
Configuration Space

- So now we have two ways of describing the system:
  1. We have the \(3n\) numbers that describe where each particle is in “normal” Cartesian space
  2. We have the \(s\) generalized coordinates

- Mathematically, we can think of each of the \(q\)’s as representing an axis in a space
  - This is called “configuration space”
  - The set of \(s\) values for the \(q\)’s gives a point in this space