Lecture 19: Forces in the Lagrangian Approach

• Hamilton’s Principle, in which only energy is mentioned, is quite different from Newton’s Laws, which depend on the concept of force
  – …and yet we seem to get to the same result no matter which method is used
• Is that coincidence? No!
• To see why not, start with Lagrange’s Equation in rectangular coordinates:

\[
\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0
\]

\[
\frac{\partial (T - U)}{\partial x_i} - \frac{d}{dt} \frac{\partial (T - U)}{\partial \dot{x}_i} = 0
\]
• In rectangular coordinates, $T$ depends only on $\dot{x}_i$ and $U$ depends only on $x_i$, so we have:

$$\frac{-\partial U}{\partial x_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i} = 0$$

$$F_i = \frac{d}{dt} \frac{\partial}{\partial \dot{x}_i} \left[ \frac{1}{2} m \sum_{j=1}^{3} \dot{x}_j^2 \right] = \frac{d}{dt} \left[ m\ddot{x}_i \right]$$

$$F_i = m\dddot{x}_i$$

• So the physics in Newton’s Laws and in the Lagrangian method are identical
  – But the ease with which problems can be solved is not!
Generalized Momenta

• In regular Cartesian coordinates, the Lagrangian for a single particle is:

\[ L = T - U = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 - U(x_i) \]

• Given this, we can readily interpret the physical significance of the quantity \( \frac{\partial L}{\partial \dot{x}_i} \):

\[ \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i = p_i \quad \text{It’s the momentum!} \]

• Since there’s nothing special about rectangular Cartesian coordinates in Lagrangian mechanics, we can define a generalized momentum associated with any generalized coordinate:

\[ p_q \equiv \frac{\partial L}{\partial \dot{q}_i} \]
• Keep in mind that, just as a generalized coordinate doesn’t have to have dimensions of length, a generalized momentum doesn’t have to have the usual units for momentum
  – If the generalized coordinate corresponds to an angle, for example, the generalized momentum associated with it will be an *angular* momentum
• With this definition of generalized momentum, Lagrange’s Equation of Motion can be written as:

\[
\frac{\partial L}{\partial q_j} - \frac{d}{dt} p_j = 0
\]

\[
\dot{p}_j = \frac{\partial L}{\partial q_j}
\]

Just like Newton’s Laws, if we call \( \frac{\partial L}{\partial q_j} \) a “generalized force”
More on Constraints

• In claiming that we could write $x$ and $\dot{x}$ as functions of $q$ and $\dot{q}$ (and time) we were making an assumption that the constraints on the system could be expressed as relations among the coordinates
  – In other words, the constraint equations are of the form

  \[ g(x_{\alpha,i}, t) = 0 \]

  No dependence on velocity!

• Such constraints are called *holonomic*, and the Lagrange Equations of Motion we’ve written are only true for this type of constraint
  – Non-holonomic constraints can be dealt with, but that’s beyond the scope of this course
Using Undetermined Multipliers

- One great advantage of the Lagrangian method, as we’ve seen, is that it allows us to solve for the motion of particles under constraints, even if we don’t know the force causing the constraint.

- In some cases, though, we’d like to determine the forces of constraint:
  - i.e., an engineer designing a mechanical device to provide a constraint needs to know how strong it must be.

- We can do this using the form of Lagrange’s equations with the constraints taken into account explicitly:

\[
\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_{k=1}^{m} \lambda_k (t) \frac{\partial g_k}{\partial q_j} = 0
\]
• Note that there are $3n$ such equations for an $n$-particle system
  – The set of Lagrange equations in which the variables are already related to each other consists of $3n-m$ equations
• In total, we have $3n+m$ equations
  – since the $m$ equations of constraint are still available to us
• This means we can solve for all the $q$’s and all the $\lambda$’s
• But what do the $\lambda$’s represent, physically?
  – Recall that they’re called “undetermined multipliers” since we don’t know the value beforehand
  – But they have the values needed to make sure the constraint is obeyed – a hint that they are closely related to the forces of constraint!
Generalized Force of Constraint

- To see what the $\lambda$’s mean, imagine that whatever part of the system that causes the constraints is removed.
- Now apply a set of external forces $Q_j$ to the system, such that the system moves as though it were constrained.
- Then the equation of motion would be:
  \[
  \dot{p}_j = \frac{\partial L}{\partial q_j} + Q_j
  \]
  \[
  \frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + Q_j = 0
  \]
- But we already have an equation of motion for the constrained system:
  \[
  \frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_{k=1}^{m} \lambda_k (t) \frac{\partial g_k}{\partial q_j} = 0
  \]
• For this to be consistent, we see that $Q_j$ must be given by:

$$Q_j = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}$$

• So finding the $\lambda$’s is a key to determining the forces of constraint.

• Of course, when I say “force”, I really mean “generalized force”.

• As with all the “generalized” quantities, some care needs to be taken in the interpretation:
  - i.e., the units won’t always be that for a force
  - for example, if the generalized coordinate represents an angle, the generalized force will be a torque.
Example: Tension in a pendulum’s string

• Given the following simple pendulum, find the tension in the string:

\[ r = \sqrt{r^2 + (l - g)^2} \]

I use the following generalized coordinates:

\[ \theta : \text{angle as shown in diagram} \]
\[ r : \text{distance from } O \text{ to } m \]

• In this case, we have a simple equation of constraint:

\[ g = r - l = 0 \]

• The kinetic and potential energies are:

\[ T = \frac{1}{2} m (r \dot{\theta})^2 + \frac{1}{2} m r^2; \quad U = -mgr \cos \theta \]
• With this, the equations of motion are:

\[
\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} + \lambda(t) \frac{\partial g}{\partial r} = 0
\]

\[
\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} + \lambda(t) \frac{\partial g}{\partial \theta} = 0
\]

• Using what we know about \( L \) and \( g \) these become:

\[
mr\dot{\theta} + mg \cos \theta - m\ddot{r} + \lambda(t)(1) = 0
\]

\[
-mgr \sin \theta - mr^2 \ddot{\theta} + \lambda(t)(0) = 0
\]

• But we know from the equation of constraint that \( r = l \) and \( \ddot{r} = 0 \) so the first equation tells us that:

\[
\lambda(t) = -mg \cos \theta = ml\dot{\theta}^2
\]

• And the tension is just the generalized force for \( r \):

\[
T = Q_r = \lambda \frac{\partial g}{\partial r} = -mg \cos \theta - ml\dot{\theta}^2
\]