Lecture 21: Hamiltonians

- We saw last time that, under certain circumstances, we can give a physical interpretation to $H$:

\[ H = 2T - L = 2T - (T - U) = T + U = E \]

- Note that the Hamiltonian is equal to the total energy only if:

1. The relationship between rectangular and generalized coordinates is independent of time ( scleronomic system )
2. The potential is independent of time

- If these conditions are met, energy is conserved as long as the Lagrangian doesn’t explicitly depend on time
  - i.e., the requirement is \( \frac{\partial L}{\partial t} = 0 \), not \( \frac{dL}{dt} = 0 \).
Hamiltonian Dynamics

• We have already introduced the generalized momentum corresponding to a given generalized coordinate:

\[ p_j = \frac{\partial L}{\partial \dot{q}_j}; \quad \dot{p}_j = \frac{\partial L}{\partial q_j} \]

• The definition of the Hamiltonian we’ve been using is:

\[ H \equiv \sum_j \left( \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} \right) - L \]

• But since we only consider potentials that don’t depend on velocities,

\[ p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \]

so we can rewrite the Hamiltonian as:

\[ H \equiv \sum_j p_j \dot{q}_j - L \]
• The fundamental difference between the Hamiltonian and Lagrangian is the variables these functions depends upon:
  – The Lagrangian is a function of $q_j, \dot{q}_j$, and $t$
  – The Hamiltonian is a function of $p_j, q_j$, and $t$

• This may not appear obvious from the definition of the Hamiltonian, in which $\dot{q}_j$ appears
  – But the $\dot{q}_j$ can themselves be expressed as functions of $q_j$ and $p_j$
Example: Hamiltonian for a Projectile

- We’ve already derived the Lagrangian for a projectile:

\[
L = \frac{1}{2} m \sum_i \dot{x}_i^2 - mgx_2
\]

\[
p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i; \quad \dot{x}_i = \frac{p_i}{m}
\]

\[
H = \sum_i p_i \dot{x}_i - L
\]

\[
= \sum_i p_i \left( \frac{p_i}{m} \right) - \frac{1}{2} m \sum_i \left( \frac{p_i}{m} \right)^2 + mgx_2
\]

\[
H(p_i, x_i, t) = \frac{1}{2m} \sum_i p_i^2 + mgx_2
\]

- Note that this problem meets the conditions under which the Hamiltonian is the total energy of the system.
• We can now arrive at equations of motion by taking the full differential of the Hamiltonian:

\[ dH(p_j, q_j, t) = \sum_j \left( \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt \]

• But we also can take the derivative starting from the definition of the Hamiltonian:

\[
\begin{align*}
\frac{dH}{dt} &= d \left( \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j, t) \right) \\
&= \sum_j \left[ \dot{q}_j dp_j + p_j dq_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right] - \frac{\partial L}{\partial t} dt \\
&= \sum_j \left[ \dot{q}_j dp_j + p_j dq_j - \dot{p}_j dq_j - p_j \dot{d}q_j \right] - \frac{\partial L}{\partial t} dt \\
&= \sum_j \left[ \dot{q}_j dp_j - \dot{p}_j dq_j \right] - \frac{\partial L}{\partial t} dt
\end{align*}
\]
• Since the two expressions for the differential must be the same, we have:

\[ \sum_j \left( \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt = \sum_j \left[ \dot{q}_j dp_j - \dot{p}_j dq_j \right] - \frac{\partial L}{\partial t} \]

• This implies that:

\[ \dot{q}_j = \frac{\partial H}{\partial p_j} \]
\[ \dot{p}_j = -\frac{\partial H}{\partial q_j} \]

Hamilton’s Equations of Motion

• Also we have:

\[ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \]
• Plugging Hamilton’s equations of motion back into the expression for $dH$ gives:

\[
\frac{dH}{dt} = \sum_j \left( \frac{\partial H}{\partial p_j} \dot{p}_j + \frac{\partial H}{\partial q_j} \dot{q}_j \right) + \frac{\partial H}{\partial t}
\]

\[
= \sum_j \left( \frac{\partial H}{\partial p_j} \left[ -\frac{\partial H}{\partial q_j} \right] + \frac{\partial H}{\partial q_j} \left[ \frac{\partial H}{\partial p_j} \right] \right) +
\]

\[
= 0 + \frac{\partial H}{\partial t}
\]

• If the Hamiltonian doesn’t depend explicitly on time, then it must be constant
Notes about Hamiltonian Dynamics

• For a system with $s$ degrees of freedom, there are $2s$ equations of motion in the Hamiltonian approach
  – i.e., for each generalized coordinate we have a equation for $\dot{q}$ and $\dot{p}$.

• However, these equations are all \textit{first-order} differential equations, whereas the $s$ Lagrange equations of motion are second order

• For most mechanics problems, there is no advantage to using the Hamiltonian method rather than the Lagrangian
  – In the general case, after all, we need to find the Lagrangian to determine the Hamiltonian

• We study Hamiltonians because they offer a very general description of mechanics, that can be extended to other areas of physics (quantum mechanics in particular!)
Cyclic Coordinates

- There is one nice feature of the Hamiltonian approach within the context of classical mechanics.

- Consider what happens when the Hamiltonian is independent of one of the coordinates.
  - Such coordinates are called cyclic.

- In the Hamiltonian approach, we see immediately that the momentum conjugate to that coordinate is a constant of the motion:
  \[
p_k = -\frac{\partial H}{\partial q_k} = 0 \text{ if } H \text{ is independent of } q_k
  \]
  \[p_k = \text{const}\]
Example: Bead on a Helix

- A bead of mass $m$ is strung on a helical wire, and acted upon by gravity
- We want to find the equations of motion using the Hamiltonian approach

Equation of helix:

\[ z = k\theta \]
\[ r = \text{const} = R \]
We start by taking \( z \) and \( \theta \) as the generalized coordinates:

\[
T = \frac{1}{2} m \left( \dot{z}^2 + R^2 \dot{\theta}^2 \right) = \frac{1}{2} m \left( \dot{z}^2 + \frac{R^2}{k^2} \frac{\dot{z}^2}{k^2} \right)
\]

\[
= \frac{1}{2} m \left( 1 + \frac{R^2}{k^2} \right) \dot{z}^2
\]

\[
U = mgz
\]

\[
L = \frac{1}{2} m \left( 1 + \frac{R^2}{k^2} \right) \dot{z}^2 - mgz
\]

\[
p_z = \frac{\partial L}{\partial \dot{z}} = m \left( 1 + \frac{R^2}{k^2} \right) \dot{z}
\]

\[
H = \dot{z} p_z - \frac{1}{2} m \left( 1 + \frac{R^2}{k^2} \right) \dot{z}^2 + mgz
\]

\[
= \frac{p_z^2}{m \left( 1 + \frac{R^2}{k^2} \right)} - \frac{1}{2} \left( \frac{p_z^2}{m \left( 1 + \frac{R^2}{k^2} \right)} \right) + mgz
\]

Note that in the Hamiltonian approach we must apply all the constraints at the beginning of the problem.
\[
H = \frac{p_z^2}{2m\left(1 + \frac{R^2}{k^2}\right)} + mgz
\]

- This case meets the conditions under which \( H = E \), so we could have taken that shortcut to find the Hamiltonian.
- Applying Hamilton’s equation of motion for \( z \) we find

\[
\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m\left(1 + \frac{R^2}{k^2}\right)}
\]

\[
\dot{p}_z = -\frac{\partial H}{\partial z} = -mg
\]

\[
\ddot{z} = \frac{\dot{p}_z}{m\left(1 + \frac{R^2}{k^2}\right)} = \frac{-g}{m\left(1 + \frac{R^2}{k^2}\right)}
\]