Lecture 26: Bounded Gravitational Orbits

• If $\varepsilon < 1$, $\varepsilon / r = 1 + \cos \theta$ is the equation of an ellipse with one focus at the origin
  – This is the general case for a bounded orbit due to gravity
• We can gain some insight into the properties of the ellipse by writing the orbital equation in Cartesian coordinates:

$$r = \frac{\alpha}{1 + \varepsilon \cos \theta}$$

$$\sqrt{x^2 + y^2} = \frac{\alpha}{1 + \varepsilon \frac{x}{\sqrt{x^2 + y^2}}}$$

$$1 = \frac{\alpha}{\sqrt{x^2 + y^2 + \varepsilon x}}$$

$$\sqrt{x^2 + y^2} = \alpha - \varepsilon x$$

$$x^2 + y^2 = \alpha^2 - 2\alpha \varepsilon x + \varepsilon^2 x^2$$
\[ x^2 (1 - \varepsilon^2) + 2\alpha\varepsilon x + y^2 = \alpha^2 \]

- Doesn’t look familiar yet, but let’s add a constant to both sides:

\[
x^2 (1 - \varepsilon^2) + 2\alpha\varepsilon x + \frac{\alpha^2 \varepsilon^2}{1 - \varepsilon^2} + y^2 = \alpha^2 + \frac{\alpha^2 \varepsilon^2}{1 - \varepsilon^2}
\]

\[
(1 - \varepsilon^2) \left[ x + \frac{\alpha\varepsilon}{1 - \varepsilon^2} \right]^2 + y^2 = \frac{\alpha^2}{1 - \varepsilon^2}
\]

\[
\frac{(1 - \varepsilon^2)^2}{\alpha^2} \left[ x + \frac{\alpha\varepsilon}{1 - \varepsilon^2} \right]^2 + \frac{1 - \varepsilon^2}{\alpha^2} y^2 = 1
\]

- Recall from geometry that the general form for the equation of an ellipse is:

\[
\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1
\]

\[ a \text{ and } b \text{ are the semi-major and semi-minor axes} \]
• Just a reminder about ellipses:
• So not only have we proven that the orbit is an ellipse, we’ve also found the lengths of the axes:

\[
a = \frac{\alpha}{1-\varepsilon^2} = \frac{l^2 / \mu k}{1-\left(1+\frac{2El^2}{\mu k^2}\right)} = \frac{l^2 / \mu k}{\frac{2El^2}{\mu k^2}} = -\frac{k}{2E}
\]

\[
b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l^2 / \mu k}{\sqrt{-\frac{2El^2}{\mu k^2}}} = \frac{l}{\sqrt{-2\mu E}}
\]

• Note that the total energy must be negative since these are bound orbits – so the minus signs in the above expressions make sense!

• Also, we see that the major axis is determined by \(E\) alone, while the minor axis depends on both \(E\) and \(l\)
Other Properties of the Orbit

• We can also readily determine the closest and furthest distances from the origin for any orbit:

\[ r = \frac{\alpha}{1 + \varepsilon \cos \theta} \]

\[ \frac{dr}{d\theta} = \frac{\varepsilon \alpha \sin \theta}{(1 + \varepsilon \cos \theta)^2} = 0 \text{ when } \theta = 0 \text{ or } \pi \]

\[ r_{\text{min}} = \frac{\alpha}{1 - \varepsilon} = a (1 - \varepsilon) \quad \text{“Pericenter”} \]

\[ r_{\text{max}} = \frac{\alpha}{1 + \varepsilon} = a (1 + \varepsilon) \quad \text{“Apocenter”} \]

• For orbits around the Earth, the terms “apogee” and “perigee” are used

• For orbits around the sun, it’s “aphelion” and “perihelion”
Orbital Period

- From conservation of angular momentum that the areal velocity is constant, and given by:
  \[ \frac{dA}{dt} = \frac{l}{2\mu} \]

- Therefore the time it takes to travel all the way around an elliptical orbit is:
  \[ \tau = \frac{A}{dA} = \frac{\pi ab}{l} = \frac{2\pi \mu ab}{l} \]

- In terms of \( E, l, \mu, \) and \( k, \) this is:
  \[ \tau = \frac{2\pi \mu}{2|E|\sqrt{2\mu|E|}} \cdot \frac{l}{l} = \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2} \]
• We can also write the period as:

\[ \tau = \frac{2\pi\mu ab}{l} = \frac{2\pi\mu a\sqrt{\alpha a}}{l} = \frac{2\pi a^{1/2}\mu}{l} a^{3/2} \]

\[ \tau^2 = \frac{4\pi^2 \alpha \mu^2}{l^2} a^3 = \frac{4\pi^2 \left(\frac{l^2}{\mu k}\right) \mu^2}{l^2} a^{3/2} = 4\pi^2 \frac{\mu}{k} a^3 \]

• If the planet’s mass is small compared to the Sun, we have:

\[ \tau^2 \approx 4\pi^2 \frac{m}{GM_{\text{sun}} m} a^3 = \frac{4\pi^2}{GM_{\text{sun}}} a^3 \]

• With this, we’ve now proven all three of Kepler’s Laws of planetary motion:

1. Planets move in elliptical orbits with the sun at one focus
2. The areal velocity is constant
3. The square of the period is proportional to the cube of the major axis of the orbit
Orbital Dynamics

- We now consider what is involved in changing a satellite’s orbit
  - For example, a mission to Mars requires taking a spacecraft from Earth’s orbit and placing it in Mars’ orbit
- For simplicity, we’ll assume that both the Earth and Mars have circular orbits
  - Not a terrible approximation: $\varepsilon = 0.02$ for the Earth and $0.09$ for Mars
- It turns out the the most energy-efficient way of doing this is known as a *Hohmann transfer*
Hohmann Transfer

• This transfer consists of two velocity changes (think of them as two fast rocket burns), both tangential to the velocity:

1. The first burn changes $E$ and $l$, turning the initial circular orbit into an elliptical orbit with apocenter equal to the final orbit’s radius
   • This elliptical orbit is called the transfer orbit

2. When the spacecraft reaches the apocenter of the transfer orbit, a second burn changes the orbit into a circular one

• The energy and velocity of the initial orbit are:

\[
E = -\frac{k}{2r_1} = T + U = \frac{1}{2}mv_1^2 - \frac{k}{r_1}
\]

\[
v_1 = \sqrt{\frac{k}{mr_1}}
\]
• For the transfer orbit, the major axis must be the sum of the radii of the initial and final circular orbits:

\[ 2a_t = r_1 + r_2 \]

\[ a_t = \frac{r_1 + r_2}{2} \]

• Therefore the energy of the transfer orbit must be:

\[ E = -\frac{k}{2\left(\frac{r_1 + r_2}{2}\right)} = -\frac{k}{r_1 + r_2} \]

• With this, we can find the velocity just after the first rocket burn:

\[ E = -\frac{k}{r_1 + r_2} = T + U = \frac{1}{2}mv_{t1}^2 - \frac{k}{r_1} \]

\[ v_{t1}^2 = \frac{2}{m}\left[\frac{k}{r_1} - \frac{k}{r_1 + r_2}\right] = \frac{2k}{m}\left[\frac{r_2}{r_1(r_1 + r_2)}\right] \]
\[ v_{t1} = \sqrt{\frac{2k}{m} \left[ \frac{r_2}{r_1 (r_1 + r_2)} \right]} \]

- The initial velocity change needed is
\[
\Delta v_1 = v_{1t} - v_1
\]

- We can also find the velocity of the transfer orbit when it reaches \( r_2 \):
\[
- \frac{k}{r_1 + r_2} = \frac{1}{2} m v_{t2}^2 - \frac{k}{r_1 + r_2} = \frac{1}{2} m v_{t2}^2 - \frac{k}{r_2}
\]

\[ v_{t2} = \sqrt{\frac{2}{m} \left[ \frac{k}{r_2} - \frac{k}{r_1 + r_2} \right]} = \sqrt{\frac{2k}{m} \left[ \frac{r_1}{r_2 (r_1 + r_2)} \right]} = \sqrt{\frac{2k}{m} \left[ \frac{r_1}{r_2 (r_1 + r_2)} \right]}
\]

- While we know a circular orbit at \( r_2 \) must have velocity
\[
v_2 = \sqrt{\frac{k}{mr_2}}
\]
• Which means that the second rocket burn must add a velocity:
\[ \Delta v_2 = v_2 - v_{2t} \]

• The time the transfer will take is given by half of the period of the transfer orbit. For Kepler’s Third Law, this is:
\[ T_t = \pi \sqrt{\frac{m}{k}} a_t^{3/2} = \pi \sqrt{\frac{1}{GM_{\text{sun}}}} a_t^{3/2} \]

• From here to Mars, that’s:
\[ T_t = \pi \sqrt{\frac{1}{(6.67 \times 10^{-11} \text{ Nm}^2 / \text{kg}^2)(2 \times 10^{30} \text{ kg})}} (1.9 \times 10^{11} \text{ m})^{3/2} \]
\[ = 2.2 \times 10^7 \text{ s} = 0.7 \text{ yr} \]

• It would take over 44 years to get to Pluto this way!