Lecture 3: Vectors

• Any set of numbers that transform under a rotation the same way that a point in space does is called a vector
  – i.e.,
  \[ A'_i = \sum_j \lambda_{ij} A_j \]
  – In earlier courses, you may have learned that a vector is, basically, an arrow
  – That’s true in three dimensions, but this new definition allows one to create higher-dimensional vectors
  – We’ll also define other objects (e.g., tensors) in terms of how they transform under rotations

• In addition to position, we’ll be using velocity, acceleration, and momentum vectors throughout this course
Vector Operations

• Adding two vectors:
  \[ \mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k} \]
  – Result is also a vector

• Multiplying a vector by a scalar:
  \[ c\mathbf{A} = cA_1\mathbf{i} + cA_2\mathbf{j} + cA_3\mathbf{k} \]
  – Again, the result is a vector

• Multiplying two vectors:
  – Here there are two options
    – One is the dot (or scalar) product:
      \[ \mathbf{A} \cdot \mathbf{B} = \sum_k A_k B_k = |\mathbf{A}| |\mathbf{B}| \cos \theta \]
      – Result is a scalar, as name suggests
      – Handy way to find angle between two vectors
Cross (or Vector) Product

• For the special case of three-dimensional vectors, we can also define a multiplication that results in another vector

\[(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k} \varepsilon_{ijk} A_j B_k\]

• The symbol \(\varepsilon_{ijk}\) is defined as:
  – 0, if any two indices are the same
  – +1, if the indices are in the order 1,2,3 or any cyclic permutation
  – -1, if the indices are not in a cyclic permutation of 1,2,3

• For example:

\[\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1; \ \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1\]

• Believe it or not, we’ll find this symbol useful throughout the course
• The cross product results in a vector perpendicular to the two vectors that are multiplied, with length $|A||B|\sin \theta$
  
  – Use right-hand rule to get direction:

Note that $A \times B = -B \times A$

Useful for finding direction perpendicular to two vectors

• Can also use matrix determinant to find cross product

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Cross product is only defined in three-dimensional space
Vector Calculus

• A lot of what we’ll be doing involves integration or differentiation of vectors

• The first thing we need to define is differentiation with respect to a scalar:

\[ \frac{dA}{ds} = \frac{dA_1}{ds} \mathbf{i} + \frac{dA_2}{ds} \mathbf{j} + \frac{dA_3}{ds} \mathbf{k} \]

  – The result is another vector

• We use this quite often in physics:

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \]

In fact, we differentiate with respect to time so often, we define a shorthand notation:

\[ \frac{dA}{dt} \equiv \dot{A}, \quad \frac{d^2A}{dt^2} \equiv \ddot{A} \]
Other Coordinate Systems

- Sometimes the symmetry of a problem will make it more convenient to work in non-cartesian coordinates.
- Defining the velocity and acceleration vectors in other coordinates gets tricky.
  - Since one needs to account for the fact that the \( r \) direction, for example, is changing with time.
- As an example, the velocity vector in spherical coordinates is:
  \[ \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta \dot{\varphi}\mathbf{e}_\varphi \]
  - Acceleration is an even more complex expression.
- We’ll learn techniques later that allow us to easily adopt convenient coordinate systems.
Gradient

• There are also cases where one can differentiate a scalar, and get a vector result.
• Consider a scalar function \( h(x_1, x_2) \) that describes a surface in three dimensions.
• Now let’s say one moves a distance \( dx_1, dx_2 \)
  – Can describe this by a vector: \( dr = dx_1 \mathbf{i} + dx_2 \mathbf{j} \)
• The change in \( h \) is given by:
  \[
  dh = \frac{\partial h}{\partial x_1} dx_1 + \frac{\partial h}{\partial x_2} dx_2
  \]
• Note that this looks like something dotted with \( dr \):
  \[
  dh = \mathbf{A} \cdot dr
  \]
  \[
  \mathbf{A} = \frac{\partial h}{\partial x_1} \mathbf{i} + \frac{\partial h}{\partial x_2} \mathbf{j}
  \]
• A is called the gradient of \( h \), and denoted by \( \nabla h \)

• We can determine its physical significance as follows:
  – Assume we’re moving along a line of constant \( h \)
    • That is, \( dh = 0 \)
  
  – So,

\[
(\nabla h) \cdot dr = 0
\]

  – But we know that \( dr \) isn’t 0 (since \( x_1 \) and \( x_2 \) are changing)
  
  – Also, \( \nabla h \) is in general not zero
  
  – This means that the only way for \( dh \) to be zero is if \( dr \) and \( \nabla h \)
    are perpendicular

• Which means that \( \nabla h \) is along the direction in which \( h \)
  changes most rapidly
• The $\nabla$ operator can be used in other ways as well, to define:
  
  – The *divergence* of a vector field:

  \[
  \text{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}
  \]

  – The *curl* of a vector field:

  \[
  \text{curl}\, \mathbf{A} = \nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \\ \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \\ \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \end{pmatrix} \mathbf{k}
  \]

• Note that the divergence is a scalar, while the curl is a vector
• If one applies the gradient operator twice, the result is called the \textit{Laplacian}:

\[
\nabla \cdot \nabla = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} = \sum_i \frac{\partial^2}{\partial x_i^2}
\]

• The Laplacian is denoted $\nabla^2$, and typically acts on a scalar function:

\[
\nabla^2 \varphi = \sum_i \frac{\partial^2 \varphi}{\partial x_i^2}
\]

• If you are taking electromagnetism, you’ll see this all the time!
Vector Integration

- One type of vector integral we’ll encounter is:
  \[ \int A d\xi \]

  where \( \xi \) is any variable we’re interested in

- This integral results in a \textit{vector}:
  \[ \int A d\xi = \int A_1 d\xi \mathbf{i} + \int A_2 d\xi \mathbf{j} + \int A_3 d\xi \mathbf{k} \]

  - i.e., it’s really a shorthand way of writing \textit{three} regular integrals

- Sometimes we’ll integrate a vector over a volume:
  \[ \int_v A dv = \iiint A_1 dx dy dz \mathbf{i} + \iiint A_2 dx dy dz \mathbf{j} + \iiint A_3 dx dy dz \mathbf{k} \]

  - In this case we need to do \textit{nine} integrals (but often symmetry will make some of these trivial)
Another form of vector integral we’ll encounter is the line (or *path*) integral:

\[ \int_{r_1}^{r_2} A \cdot ds \]

- This means that one moves from the initial point \( r_1 \) to the final point \( r_2 \) along a defined path, and sums the component of \( A \) parallel to that path at each point:

One example from mechanics is the definition of *work*:

\[ W = \int_{r_1}^{r_2} F \cdot ds \]

- In this case, the integral result is a *scalar*
• Similarly, one can define an integral over a surface:

\[ \int_S \mathbf{A} \cdot d\mathbf{a} \]

  – This is really a double-integral over both dimensions of the surface

• The vector \( d\mathbf{a} \) has length equal to a small piece of the surface area (i.e., \( dxdy \)), and has direction perpendicular to the surface

  – For closed surfaces, we take the direction to be outward

• As with the line integral, this will result in a scalar
Gauss’ and Stokes’ Theorems

• Two distinguished mathematicians discovered some interesting relationships among vector integrals
• When integrating over a closed surface, we can use Gauss’ theorem:
  \[ \int_S \mathbf{A} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{A}\,dv \]
  – \( V \) is the volume enclosed by the surface in the left-hand integral
• When integrating around a closed path (i.e. a loop), we can use Stokes’ theorem:
  \[ \int_C \mathbf{A} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} \]
  – \( S \) is an open surface, with the open edge bounded by the loop \( C \)
• Gauss’ and Stokes’ theorems can be used to reduce the dimensionality of some integrals