Lecture 31: Kinematics of Elastic Collisions

• Last lecture we found that elastic collisions are easy to analyze in the CM frame. We also found the following relations between CM and LAB frame speeds:

\[
\frac{m_1 u_1}{m_1 + m_2} = V_{CM}
\]

\[
v_1' = u_2' = V_{CM} = \frac{m_1 u_1}{m_1 + m_2}
\]

\[
v_1' = u_1' = \frac{m_2 u_1}{m_1 + m_2}
\]
• We now focus on the motion of $m_1$, using our knowledge of non-relativistic (aka Galilean) velocity transformations:
  – Velocity components perpendicular to the boost direction are constant (this is true in relativity, too)
  – Velocity components along the boost direction differ by the boost velocity
• Applying these facts to the motion of $m_1$ after the collision in both the CM and LAB frames, we find:

\[
\begin{align*}
\nu_1' \sin \theta &= \nu_1 \sin \psi \\
\nu_1' \cos \theta + V_{CM} &= \nu_1 \cos \psi
\end{align*}
\]

• Dividing these equations gives:

\[
\tan \psi = \frac{\nu_1' \sin \theta}{\nu_1' \cos \theta + V_{CM}} = \frac{\sin \theta}{\cos \theta + V_{CM} / \nu_1'}
\]
• The relationship between $V_{CM}$ and $v_1'$ is:

$$\frac{V_{CM}}{v_1'} = \frac{m_1 u_1}{m_2 u_1} = \frac{m_1}{m_2}$$

• Thus we find the relationship between the scattering angle in the LAB frame and that in the CM frame

$$\tan \psi = \frac{\sin \theta}{\cos \theta + V_{CM} / v_1'} = \frac{\sin \theta}{\cos \theta + m_1 / m_2}$$
- Plots of $\psi$ vs. $\theta$ for some values of $m_1/m_2$:

$m_1/m_2 = 0$

$m_1/m_2 = 0.5$

$m_1/m_2 = 1.0$

$m_1/m_2 = 1.5$

Note that if $m_1 > m_2$ there are two possible $\theta$ values for every $\psi$. 
Energy in Elastic Collisions

• We can also think about collisions in terms of the energy
• In particular, we’ll explore the final kinetic energy of \( m_1 \) in the LAB frame
• But to start, let’s write down the initial kinetic energy of the system, in both the LAB and CM frames:

LAB:

\[ T_0 = \frac{1}{2} m_1 u_1^2 \quad \text{True since} \quad u_2 = 0 \]

CM:

\[ T_0' = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 \]

• Using what we know about \( u_1' \) and \( v_2' \) in terms of \( u_1 \), we can rewrite this as:
This shows that the kinetic energy measured in the CM frame is smaller than that measured in any other frame. Consistent with the notion that \( T \) is the sum of energy due to motion within the system plus center-of-mass energy of the system.
The fraction of the kinetic energy retained by $m_1$ after the collision in the LAB frame is:

$$\frac{T_1}{T_0} = \frac{1}{2} \frac{m_1 v_1^2}{m_1 u_1^2} = \frac{v_1^2}{u_1^2}$$

To relate this to the LAB scattering angle, we use conservation of energy and momentum:

$$p_x = m_1 u_1 = m_1 v_1 \cos \psi + m_2 v_2 \cos \zeta$$

$$p_y = 0 = m_1 v_1 \sin \psi - m_2 v_2 \sin \zeta$$

$$T = \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$
• From the energy equation, we find:

\[ v_2 = \sqrt{\frac{m_1}{m_2}} (u_1^2 - v_1^2) \]

• Plugging this into the \( p_y \) equation gives:

\[ m_1 v_1 \sin \psi = m_2 \sqrt{\frac{m_1}{m_2}} (u_1^2 - v_1^2) \sin \zeta \]

\[ \sin \zeta = \frac{m_1 v_1 \sin \psi}{\sqrt{m_1 m_2 (u_1^2 - v_1^2)}} = \sqrt{1 - \cos^2 \zeta} \]

\[ \cos \zeta = \sqrt{1 - \frac{m_1^2 v_1^2 \sin^2 \psi}{m_1 m_2 (u_1^2 - v_1^2)}} = \sqrt{\frac{m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi}{m_2 (u_1^2 - v_1^2)}} \]
• This means that:

\[ m_2 v_2 \cos \zeta = m_2 \sqrt{\frac{m_1 (u_1^2 - v_1^2)}{m_2}} \frac{m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi}{m_2 (u_1^2 - v_1^2)} \]

\[ = \sqrt{m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right]} \]

• We can now insert this into the \( p_x \) equation to find:

\[ m_1 u_1 = m_1 v_1 \cos \psi + \sqrt{m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right]} \]

\[ m_1 (u_1 - v_1 \cos \psi) = \sqrt{m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right]} \]

\[ m_1^2 u_1^2 - 2m_1^2 v_1 \cos \psi + m_1^2 v_1^2 \cos \psi = m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right] \]
• Or:

\[ v_1^2 (m_1 + m_2) - 2m_1 u_1 v_1 \cos \psi + u_1^2 (m_1 - m_2) = 0 \]

\[ \frac{v_1^2}{u_1^2} (m_1 + m_2) - 2m_1 \frac{v_1}{u_1} \cos \psi + m_1 - m_2 = 0 \]

from which we can find \( v_1 / u_1 \) with the quadratic formula:

\[
\frac{v_1}{u_1} = \frac{2m_1 \cos \psi \pm \sqrt{4m_1^2 \cos^2 \psi - 4 \left( m_1^2 - m_2^2 \right)}}{2(m_1 + m_2)}
\]

\[
= \frac{m_1}{m_1 + m_2} \left[ \cos \psi \pm \sqrt{\cos^2 \psi - \left( 1 - \frac{m_2^2}{m_1^2} \right)} \right]
\]

\[
= \frac{m_1}{m_1 + m_2} \left[ \cos \psi \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2 \psi} \right]
\]
Inelastic Collisions

• All collisions conserve energy (just like elastic collisions do)

• But, if *systems* of particles collide, the internal energy of the systems may change
  – Therefore, the kinetic energy associated with the motion of the center of mass is not conserved

• Such collisions are called *inelastic*
  – The extreme case is a collision between two objects that stick together after they collide (two blobs of clay or silly putty might behave this way)
  – These collisions are called *totally inelastic*

• Most collisions lie somewhere between elastic and totally inelastic
Quantifying Inelasticity

• There are two ways to measure how inelastic a collision is

1. Measure the kinetic energy before and after the collision, and call the difference $Q$:

$$ T_f - T_i = Q $$

$$ Q + \frac{1}{2} m_1^2 v_1^2 + \frac{1}{2} m_2^2 v_2^2 = \frac{1}{2} m_1^2 v_1'^2 + \frac{1}{2} m_2^2 v_2'^2 $$

The larger $|Q|$ is, the more inelastic the collision

– If $Q > 0$, the final kinetic energy is greater than the initial. Such collisions are called *exoergic* – an explosion is one example

– If $Q < 0$, the final kinetic energy is less than the initial. These collisions are *endoergic* – a collision between blobs of clay falls into this category
Coefficient of Restitution

2. Another way to think about inelasticity is to consider the relative velocity in a head-on collision. In the CM frame:

- **Elastic collision**
  \[ \frac{|v_1 - v_2|}{|u_1 - u_2|} = 1 \]
  \[ v_1 = -u_1 \quad \text{and} \quad v_2 = -u_2 \]

- **Totally inelastic collision**
  \[ \frac{|v_1 - v_2|}{|u_1 - u_2|} = 0 \]
  \[ m_1 + m_2 \quad \text{and} \quad v_1 = v_2 = 0 \]

- In general we define \( \frac{|v_1 - v_2|}{|u_1 - u_2|} = \varepsilon \) as the coefficient of restitution.

- For head-on collisions in non-CM reference frames, the velocity components normal to the collision plane enter the formula.
Impulse

- Even though we may not know what forces act during a collision, we can determine something about those forces from Newton’s Second Law.

- During the collision:

\[ F = \dot{p} = \frac{d(mv)}{dt} \]

\[ Fdt = d(mv) \]

\[ \int_{t_1}^{t_2} Fdt = \Delta (mv) = m\Delta v \text{ if } m \text{ is constant} \]

- The quantity \( \Delta (mv) \) is called the *impulse*, and given the symbol \( P \).

- The left-hand side is related to the time-average of the force acting during the collision, so we have:

\[ F_{\text{avg}} \Delta t = P \]