Lecture 33: Rutherford’s Formula, and Rocket Motion

- For gravity and the electric force,
  \[ U(r) = \frac{k}{r} \]

- So we have:
  \[ \Delta \Theta = \int_{r_{\text{min}}}^{\infty} \frac{b/r^2 dr}{\sqrt{1 - k/rT_o - (b^2/r^2)}} = \int_{r_{\text{min}}}^{\infty} \frac{b/rdr}{\sqrt{r^2 - kr/T_o - b^2}} \]

- Note that we can determine \( r_{\text{min}} \) by finding the distance at which the total energy equals the effective potential.

- This integral is exactly the same as the one we did in determining the orbits of planets under gravity. Rather than do it again, I’ll just give the answer:
  \[ \cos \Delta \Theta = \frac{(\kappa/b)}{\sqrt{1 + (\kappa/b)^2}} ; \quad \kappa = \frac{k}{2T_o} \]
• Solving for $b$ gives:

\[
\left[1 + \left(\kappa/b\right)^2 \right] \cos^2 \Delta \Theta = \left(\kappa/b\right)^2
\]

\[
\left(\kappa/b\right)^2 \left[1 - \cos^2 \Delta \Theta\right] = \cos^2 \Delta \Theta
\]

\[
\left(\kappa/b\right)^2 = \frac{\cos^2 \Delta \Theta}{\sin^2 \Delta \Theta} = \cot^2 \Delta \Theta
\]

\[b = \kappa \tan \Delta \Theta\]

• Therefore,

\[
\frac{db}{d\theta} = \frac{db}{d\Delta \Theta} \frac{d\Delta \Theta}{d\theta} = \kappa \sec^2 \Delta \Theta \left(-\frac{1}{2}\right)
\]

\[= -\frac{\kappa}{2 \cos^2 \Delta \Theta} = -\frac{\kappa}{2 \cos^2 (\pi/2 - \theta/2)}\]

\[= -\frac{\kappa}{2 \sin^2 (\theta/2)}\]

Note that this is negative, as we expect
• New we’re set to find the differential cross section:

\[ \sigma(\theta) = \frac{b}{\sin \theta} \left| \frac{\kappa}{2 \sin^2 (\theta/2)} \right| = \frac{\kappa(\kappa \tan \Delta \Theta)}{2 \sin \theta \sin^2 (\theta/2)} \]

\[ = \frac{\kappa^2 \cot (\theta/2)}{2 \sin^2 (\theta/2)(2 \sin (\theta/2) \cos (\theta/2))} \]

\[ = \frac{\kappa^2}{4 \sin^4 (\theta/2)} \]

• This formula was derived by Rutherford in 1911 (classical mechanics in the 20th century!)

• By coincidence, the result also holds true for a quantum mechanical calculation (not true for other types of forces)
Total Cross Section

- Sometimes we want to know the total probability for a collision to occur when we fire beams of particles at a target (or at each other). The total cross section is determined by integrating over the differential cross section:

\[ \sigma_T = \int \sigma(\theta) \, d\Omega' = 2\pi \int \sigma(\theta) \sin \theta \, d\theta \]

- For \( k/r^2 \) forces we have:

\[
\sigma_T = \frac{\pi \kappa^2}{2} \int_0^\pi \frac{\sin \theta}{\sin^4(\theta/2)} \, d\theta = \pi \kappa^2 \int_0^\pi \frac{\sin(\theta/2) \cos(\theta/2)}{\sin^4(\theta/2)} \, d\theta \\
= \pi \kappa^2 \int_0^\pi \frac{\cos(\theta/2)}{\sin^3(\theta/2)} \, d\theta = \pi \kappa^2 \int_0^1 \frac{2du}{u^3}; \quad u = \sin(\theta/2) \\
= -\pi \kappa^2 \left[ \frac{1}{u^2} \right]_0^1 = \infty !
\]

But most of this infinite cross section results in scattering at very small angles.
Rocket Motion

• The next system we’ll study is one in which mass is ejected from one part of the system to change the motion of the remainder of the system
  – A rocket is an example of such a system
• We’ll first consider the motion of a rocket in free space – that is, with no external forces acting on it
  – Since there are no external forces, the momentum of the system (rocket plus fuel) must be constant
• Let’s compare the momentum at a given time with the momentum a small time $dt$ later:

\[
\begin{align*}
\text{Initial:} & \quad mv \\
\text{Final:} & \quad (m - dm')(v + dv) + v_em'
\end{align*}
\]

- Mass of rocket + fuel
- Velocity of exhausted fuel
- Mass of fuel exhausted during $dt$
• Rocket engines are typically designed to eject fuel at a constant speed $u$ (with respect to the rocket). This means that:

$$v_e = v - u$$

• So conservation of momentum now tells us that:

$$mv = (m - dm') (v + dv) + (v - u) dm'$$

$$= mv - vdm' + mdv - dm' dv + vdm' - u dm'$$

$$mdv - u dm' = 0$$

$$mdv = u dm'$$

$$m \frac{dv}{dt} = u \frac{dm'}{dt}$$

• This is called the “thrust” of the engine

• In terms of the change in mass of the rocket, we have:

$$m \frac{dv}{dt} = -u \frac{dm}{dt}$$
Integrating the Equation of Motion

- Since the equation of motion is separable, we can solve it to find:

\[ m \, dv = -u \, dm \]

\[ dv = -u \, \frac{dm}{m} \]

\[ \int_{v_i}^{v_f} dv = -u \int_{m_i}^{m_f} \frac{dm}{m} \]

\[ \Delta v = -u \ln \frac{m_i}{m_f} \]

To maximize final speed:
1. Eject fuel at highest speed possible
2. Minimize ratio of initial to final masses
Engineering Issues

• There are practical limits to both the speed of fuel exhaust and the ratio of initial to final masses
  – Higher exhaust speeds mean higher temperatures – limit reached when engine melts!
  – Must build the fuel tank sturdy enough to support the initial fuel weight

• A multistage rocket offers a way to achieve higher final velocity:

\[ m_o : \text{Initial mass of rocket+fuel} \]
\[ m_1 : \text{Mass of rocket after first-stage fuel gone} \]
\[ m_a : \text{Initial mass of second stage (incl. fuel)} \]
\[ m_2 : \text{Mass after second-stage fuel gone} \]

Key point: \( m_a < m_1 \) since first-stage fuel tank is discarded
• After the first stage uses up its fuel, the speed of the rocket is:

\[ v_1 = v_o + u \ln \frac{m_o}{m_1} \]

• The first-stage fuel tank is then released (note that this doesn’t change the velocity of the rocket), and second stage fires. At the end:

\[ v_2 = v_1 + u \ln \frac{m_a}{m_2} = v_o + u \ln \frac{m_o}{m_1} + u \ln \frac{m_a}{m_2} \]

\[ = v_1 + u \left( \ln \frac{m_o}{m_1} + \ln \frac{m_a}{m_2} \right) \]

\[ = v_1 + u \ln \frac{m_o m_a}{m_1 m_2} \]

Larger than single-stage version:

\[ v_1 + u \ln \frac{m_o}{m_2} \]
Blast Off

- In addition to just flying in free space, we want to use rockets to lift off from the ground.
- In this case, the external force of gravity is acting on the rocket.
  - We’ll assume that the rocket hasn’t flown very high, so this can be approximated as a constant force.
- Newton’s Second Law tells us that:

\[ F_g = \frac{dp}{dt} \]
\[ -mgdt = dp \]

- \( dp \) is the same as we calculated before, so:

\[ -mgdt = m\dot{v} + udm \]
\[ -mg = m\dot{v} + um \]
• Assume that the rate of fuel burn is constant: $m \equiv -\alpha$
• Then the equation of motion is

$$u\alpha - mg = m\dot{v}$$

$$dv = \left(\frac{u\alpha}{m} - g\right)dt = \left(\frac{u\alpha}{m} - g\right)\left(-\frac{dm}{\alpha}\right) = \left(\frac{g}{\alpha} - \frac{u}{m}\right)dm$$

$$\int_{v_i}^{v_f} dv = \int_{m_i}^{m_f} \left(\frac{g}{\alpha} - \frac{u}{m}\right)dm$$

$$\Delta v = \frac{g}{\alpha} \Delta m + u \ln \frac{m_i}{m_f}$$

$$= -gt + u \ln \frac{m_i}{m_f}$$