Lecture 39: Motion of a Symmetric Top

• We’ll now apply the equations of motion to a symmetric top
  – Meaning an object with at least two of its principal moments of inertia equal
• We’ll choose $I_1 = I_2$ (i.e., the $z$ axis is the symmetry axis) and start with the case where no external forces act on the body
• The equations of motion then become:

$$
(I_1 - I_1) \omega_1 \omega_2 - I_3 \dot{\omega}_3 = 0 \\
(I_3 - I_1) \omega_3 \omega_1 - I_1 \dot{\omega}_2 = 0 \\
(I_1 - I_3) \omega_2 \omega_3 - I_1 \dot{\omega}_1 = 0
$$

• The first equation tells us that:

$$I_3 \dot{\omega}_3 = 0, \text{ so } \omega_3 \text{ is a constant}$$
• We can now rewrite the other two equations as:

\[
\dot{\omega}_1 = -\left( \frac{I_3 - I_1}{I_1} \omega_3 \right) \omega_2 \\
\dot{\omega}_2 = \left( \frac{I_3 - I_1}{I_1} \omega_3 \right) \omega_1
\]

• The term in parentheses is constant; for simplicity we write it as \( \Omega \)

• We’re left with the following coupled differential equations:

\[
\dot{\omega}_1 = -\Omega \omega_2 \\
\dot{\omega}_2 = \Omega \omega_1
\]
• To solve the equations, we multiply the second by $i$ and add it to the first:

\[
\begin{align*}
\dot{\omega}_1 + i\dot{\omega}_2 &= -\Omega \omega_2 + i\Omega \omega_1 \\
\dot{\omega}_1 + i\dot{\omega}_2 + \Omega \omega_2 - i\Omega \omega_1 &= 0 \\
\dot{\omega}_1 + i\dot{\omega}_2 - i^2\Omega \omega_2 - i\Omega \omega_1 &= 0 \\
\dot{\omega}_1 + i\dot{\omega}_2 - i\Omega(\omega_1 + i\omega_2) &= 0
\end{align*}
\]

• We now define a new variable 

\[\eta \equiv \omega_1 + i\omega_2\]

so the equation becomes:

\[\dot{\eta} - i\Omega \eta = 0\]

• That’s an equation we know how to solve:

\[\eta = Ae^{i(\Omega t + \delta)}\]

• To make life easy, we’ll choose the phase $\delta$ to be zero
• Writing the equation in terms of the components of $\omega$ gives:

$$\omega_1 + i\omega_2 = Ae^{i\Omega t}$$

• Writing the real and imaginary parts of the above equation separately gives:

$$\omega_1 = A\cos(\Omega t)$$
$$\omega_2 = A\sin(\Omega t)$$

• To interpret this result, note that the magnitude of the angular velocity vector must be constant:

$$|\omega| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 \cos^2(\Omega t) + A^2 \sin^2(\Omega t) + \omega_3^2}$$

$$= \sqrt{A^2 + \omega_3^2} = \text{const}$$
• In other words, in the body frame the motion would look like:

![Diagram of conical motion](image)

Body cone

Symmetry axis

• How does the motion look in an inertial frame?
  – We start by noting that both \( \mathbf{L} \) and rotational kinetic energy are constant

• So, \( \mathbf{L} \) is pointed along a fixed direction in the inertial frame, which we can take to be the \( x_3' \) direction

This conical motion of the angular velocity is called “precession”
• The kinetic energy can be expressed as $T = \frac{1}{2} \mathbf{\omega} \cdot \mathbf{L}$, which implies that the component of $\mathbf{\omega}$ along the $\mathbf{L}$ direction must be constant.

• In other words, the motion must look like:

• Further, we can show that $\mathbf{L}$, $\mathbf{\omega}$, and the $x_3$ axis all must lie in the same plane:
  – First, note that: $\mathbf{\omega} \times \mathbf{e}_3 = \omega_2 \mathbf{e}_1 - \omega_1 \mathbf{e}_2 = \mathbf{A}$
  – $\mathbf{A}$ is a vector perpendicular to the plane of $\mathbf{\omega}$ and $\mathbf{e}_3$
  $\mathbf{L} \cdot \mathbf{A} = L_1 A_1 + L_2 A_2 = I_1 \omega_1 \omega_2 - I_2 \omega_2 \omega_1 = 0$ (since $I_1 = I_2$)
• Combining all of this information, we see that the geometry must be:

![Diagram of cones](image)

- The body cone can be thought of as rolling without slipping on the surface of the space cone.
Motion of a Top Under Gravity, with One Fixed Point

• We now consider the motion observed with a child’s toy top

• Again we assume there is an axis of symmetry, such that

\[ I_1 = I_2 \]

• Take the fixed point to be the origin of both the inertial and body reference frames

• Assume the center of mass of the top is a distance \( h \) from the fixed point
  - So the potential energy is
  \[ U = Mgh \cos \theta \]
  - The kinetic energy about the fixed point is purely rotational:
  \[ T = \frac{1}{2} \sum_i I_i \omega_i^2 \]
• So the Lagrangian is simply:

\[ L = \frac{1}{2} \sum_i I_i \omega_i^2 - Mgh \cos \theta \]

• But we need to express this in terms of our generalized coordinates – the Eulerian angles
  – Luckily, we already know how to write the \( \omega_i \) in terms of \( \theta, \phi, \) and \( \psi \) (and their time derivatives)

• After a little algebra (see p. 457 of the text) we find:

\[ L = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgh \cos \theta \]

• The first thing to note is that neither \( \phi \) nor \( \psi \) appear in the Lagrangian
  – So the momenta conjugate to these quantities must be conserved
\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \left( I_1 \sin^2 \theta + I_3 \cos^2 \theta \right) \dot{\phi} + I_3 \dot{\psi} \cos \theta = \text{const} \]

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) = I_3 \omega_3 = \text{const} \]

- Solving the above for \( \dot{\phi} \) gives:
  \[ \dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta} \]

- Precession again (this time about the vertical direction)
Motion in $\theta$

- The motion in $\theta$ is the most counterintuitive feature of a top’s motion
  - i.e., the top doesn’t fall over!

- We can see why simply by considering conserved quantities. First, energy is conserved (in the real world there’s friction, so tops eventually slow down, but we’re ignoring that):

$$E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 + Mgh \cos \theta$$

$$= \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + Mgh \cos \theta = \text{const}$$

- But,

$$I_3 \omega_3^2 = I_3 \left( \frac{p_{\psi}}{I_3} \right)^2 = \frac{p_{\psi}^2}{I_3} = \text{const.}$$
• Thus, the quantity:

\[ E' = \frac{1}{2} I_1 \left( \phi^2 \sin^2 \theta + \dot{\theta}^2 \right) + Mgh \cos \theta \]

\[ = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + Mgh \cos \theta \]

is also constant

• We can write this as:

\[ E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta) \]

• This looks like the expression for energy in one dimension, with \( V(\theta) \) playing the role of an “effective potential”
• A “typical” $V(\theta)$ looks like:

- For a given $E'$ there is an allowed region in $\theta$
- The body will oscillate within the allowed range
  – This oscillation is called nutation