Lecture 40: Stability of Rigid-Body Rotations

• We start with any rigid body
  – i.e., no longer assuming that two on the principal moments are the same. Let’s label the principal moment in order of their size: $I_1 < I_2 < I_3$

• Assume that in the initial state the body is rotating about the principal axes with smallest moment
  \[ \omega = \omega_1 e_1 \]

• We want to explore what happens when the body is given a slight nudge
  – Aside from this nudge, we’ll assume no external forces act on the body

• Immediately after the nudge, the angular velocity vector is:
  \[ \omega = \omega_1 e_1 + \lambda e_2 + \mu e_3 \quad \lambda \text{ and } \mu \text{ are small} \]
Applying Euler’s Equations for force-free motion to this situation, we find:

\[
\begin{align*}
(I_2 - I_3) \lambda \mu - I_1 \dot{\omega}_1 &= 0 \\
(I_3 - I_1) \mu \omega_1 - I_2 \dot{\lambda} &= 0 \\
(I_1 - I_2) \lambda \omega_1 - I_3 \dot{\mu} &= 0
\end{align*}
\]

The product \( \mu \lambda \) in the first equation is negligibly small, so \( \omega_1 \) is constant

Now consider how \( \mu \) and \( \lambda \) change with time
  - If they remain small, the motion will look pretty much like the initial state – rotation about the \( x_1 \) axis
  - If they tend to large values, the motion will come look significantly different from the initial motion

In other words, the motion is either stable or unstable
• Rearranging the second two Euler Equations gives:

\[
\dot{\lambda} = \left( \frac{I_3 - I_1}{I_2} \omega_1 \right) \mu
\]

\[
\dot{\mu} = \left( \frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda
\]

• To solve this system, take the time derivative of the first equation:

\[
\ddot{\lambda} = \left( \frac{I_3 - I_1}{I_2} \omega_1 \right) \dot{\mu}
\]

• Now substitute the expression for \( \dot{\mu} \) from above:

\[
\ddot{\lambda} = -\left( \frac{I_1 - I_3}{I_2} \omega_1 \right) \left( \frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda
\]
• Everything on the right-hand side in front of the $\lambda$ is a constant (which we define as $-\Omega^2$), so we have:

$$\ddot{\lambda} = -\Omega^2 \lambda; \quad \Omega = \omega_1 \sqrt{\left(\frac{I_1 - I_3}{I_2}\right) \left(\frac{I_1 - I_2}{I_3}\right)}$$

• The general solution to this equation is:

$$\lambda(t) = A e^{i\Omega t} + B e^{-i\Omega t}$$

we’d find a similar result for $\mu$

• This means that the time evolution of $\lambda$ depends on whether $\Omega$ is imaginary or real:

1. If $\Omega$ is real: solution describes oscillatory motion – $\lambda$ never becomes large. Rotation is stable.

2. If $\Omega$ is imaginary: the second term represents exponential growth in $\lambda$. Rotation is unstable
• Which case applies here? We assumed from the start that the $x_1$ axis had the smallest principal moment

• Therefore, both $I_1 - I_3$ and $I_1 - I_2$ are negative
  – The product is positive, and therefore $\Omega$ is real
  – So the rotation is stable!

• If the $x_1$ axis had the *largest* principal moment, both $I_1 - I_3$ and $I_1 - I_2$ would be positive
  – Again, stable rotation

• On the other hand, if the $x_1$ axis has the intermediate moment of inertia, the product $(I_1 - I_3)(I_1 - I_2)$ must be negative
  – In this case the rotation is unstable
What if two of the principal moments are equal?

• Let’s say $I_1 = I_2$. Then the Euler equations would be:

$$\dot{\lambda} = \left( \frac{I_3 - I_1}{I_2} \omega_1 \right) \mu$$

$$\dot{\mu} = \left( \frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda = 0$$

so $\mu$ is a constant. That means we can directly integrate the differential equation for $\lambda$ to find:

$$\lambda(t) = C + Dt$$

• So $|\lambda|$ becomes large, meaning rotation about $x_1$ is unstable
  – Same result is obtained for rotation about $x_2$
• Only rotations about the non-equal (symmetry) axis are stable