Lecture 41: Highlights

• The goal of this lecture is to remind you of some of the key points that we’ve covered this semester
  – Note that this is not the complete set of topics that may appear on the final exam!

• The fundamental basis for classical mechanics is provided by Newton’s Laws:
  I. A body remains at rest, or moving at constant velocity, unless acted upon by a force
  II. When a force does act on a body, the rate of change of momentum is equal to the force
  III. When two bodies exert forces on each other, the forces are equal in magnitude and opposite in direction
• Note that Newton’s Laws are not valid in all reference frames
  – Any reference frame in which the First Law holds is called an *inertial frame*.
  – Once an inertial frame is found, the other two Laws are guaranteed to be valid in that frame.
  – Any frame moving at constant velocity with respect to an inertial frame is also an inertial frame

• You should be able to:
  – Identify the forces acting on various components of a mechanical system
  – Apply Newton’s Laws to determine the Equations of Motion
Conserved Quantities

• Often, mechanical problem can be solved more easily by noting that some quantities are conserved
  – For example, the momentum of any mechanical system on which no external force acts is conserved
  – Also, the angular momentum of any system on which no external torque acts is conserved
• Both of these conserved quantities are vectors
• There is also an important scalar that is sometimes conserved: energy
  – Make sure you understand the rule for when energy is conserved: Energy is conserved if only conservative forces do work on a system
• The rule for energy conservation only makes sense if we define “work” and “conservative force”:

• Work is the quantity:

\[ W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} \]

– For example, any force that acts perpendicular to the direction of motion does no work

– Work-energy theorem: the total work done on a system equals the change in kinetic energy:

\[ W_{tot} = \Delta T = \Delta \left( \frac{1}{2}mv^2 \right) \]

• A conservative force is one that can be written as the (negative of) the gradient of a scalar:

\[ \mathbf{F} = -\nabla U \]

– \( U \) is the potential energy associated with the force
More on Energy

• Make sure you understand how to interpret the following type of plot (for one-particle system in one dimension):

For each energy, should be able to find allowed regions for particle, points with max/min kinetic energy, points with max/min net force.
Equilibrium and Oscillations

- A system is in *equilibrium* if there is no net force acting on it.
- If all the forces involved are conservative, this implies that:
  \[ \Delta U = 0 \]
  - i.e., the potential energy is at a local maximum or minimum.
- Equilibria can be stable or unstable.
  - If the equilibrium is at a local maximum of potential energy, it’s unstable. If it’s at a local minimum, it’s stable.
  - Check by examining sign of
    \[ \nabla^2 U \left( = \frac{d^2 U}{dx^2} \text{ in 1-D} \right) \]
• If a system is disturbed slightly from a stable equilibrium point, it will oscillate.

• The most common type of oscillation is simple harmonic, in which the force pushing the system back toward equilibrium increases linearly with the distance from equilibrium:

\[ F = -kx, \quad k = \frac{d^2U}{dx^2} \]

• The solution to the above equation is:

\[ x(t) = A \cos\left(\sqrt{\frac{k}{m}}t - \delta\right) \equiv A \cos(\omega_0 t - \delta) \]

\(A\) and \(\delta\) are determined by initial conditions.
• In general, damping and driving forces may also be applied to an oscillating system. If the driving force is sinusoidal, the equation of motion is:

\[ \ddot{x} + 2 \beta \dot{x} + \omega_0^2 x = A \cos \omega t \]

• This has a complementary (transient) solution…

\[ x_c(t) = e^{-\beta t} \left[ A_1 e^{\sqrt{\beta^2 - \omega_0^2} t} + A_2 e^{\sqrt{\beta^2 - \omega_0^2} t} \right] \]

– note that this behavior depends on whether the term under the square root sign is positive (overdamped), zero (critically damped), or negative (underdamped)

…and a particular solution:

\[ x_p(t) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \cos(\omega t + \delta) \]

• Coefficient may become large for some \( \omega \to \) resonance
Gravitation

- The key fact is the form of the gravitational force between two particles:

\[ F = -\frac{G M m}{r^2} e_r \]

- We also defined a gravitational field \( \mathbf{g} = \frac{F_g}{m} \) and a gravitational potential \( \Phi \) such that

\[ \mathbf{g} = -\nabla \Phi \]

- The gravitational field can be calculated directly:

\[ \mathbf{g} = -G \int_V \frac{\rho(r') e_r}{r^2} dV' \]

or with Gauss’ Law:

\[ \int_S \mathbf{g} \cdot \mathbf{n} da = -4\pi G m \]

Useful for symmetric objects
Lagrangians

• We then learned a way to formulate mechanics that looked completely different from Newton’s Laws
  – but of course it isn’t!
• Defining the Lagrangian \( L \) as:

\[
L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - U(q_j, t)
\]

the equation of motion for each of the \( q \) is:

\[
\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = 0
\]

• The \( q_j \) themselves are any set of variable that describes the configuration of the system \textit{in an inertial frame}
  – We call them generalized coordinates
Constraints

• If the system is constrained to move in a certain way, we can find the generalized forces of constraint
  
  – as long as the constraints are *holonomic* (see p. 238 of text)

• The equations of constraint are called $f_k$, and Lagrange’s Equations of Motion are modified to:

\[
\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t) \frac{\partial f_k}{\partial q_j} = 0
\]

  – Note that generalized forces aren’t always forces (sometimes they’re torques, or other quantities)

• We also defined a generalized momentum associated with any generalized coordinate:

\[
p_q = \frac{\partial L}{\partial \dot{q}_j}
\]
Hamiltonian

• From Lagrange’s equation of motion, we see that:
  \[ \dot{p}_q = \frac{\partial L}{\partial q} \]

• With the generalized momenta, we can also define the Hamiltonian \( H \):
  \[ H(q, p, t) \equiv \sum_j p_j \dot{q}_j - L \]

• In some cases, \( H \) equals \( T + U \)
  – If potential is velocity-independent and generalized coordinates have no explicit time dependence

• And we have Hamilton’s Equations of Motion:
  \[ \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = \frac{\partial H}{\partial q_k} \]
Central-Force Motion

• We spent some time exploring the motion of systems with forces acting along the line between pairs of particles
  – Mostly because that’s how many forces in nature – including gravity – behave

• Only two-body systems are considered, since they can be solved exactly

• In fact, two-body systems can be treated as one-body systems by using the effective mass:

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \]

• Both energy and angular momentum are conserved:

\[ l = \mu r^2 \dot{\theta} = \text{const} \]
• Due to this, we can treat the motion as though it were one-dimensional:

\[ E = T(r, \dot{r}, \dot{\theta}) + U(r) = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\theta}^2 + U(r) \]

\[ = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \left( \frac{l}{\mu r^2} \right)^2 + U(r) \]

\[ = \frac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2 \mu r^2} + U(r) = \frac{1}{2} \mu \dot{r}^2 + V(r) \]

where \( V(r) \) is the effective potential

• We almost always define \( U(r) \) such that \( U(\infty) = 0 \)
Gravitational Orbits

- Gravitational orbits come in only four shapes
- The general expression for an orbit is:
  \[
  \frac{\alpha}{r} = 1 + \varepsilon \cos \theta
  \]
  where \( \varepsilon \) is the *eccentricity* of the orbit
- The relationship between shape, energy, and eccentricity is given below:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Energy</th>
<th>( \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbola</td>
<td>( &gt; 0 )</td>
<td>( &gt; 1 )</td>
</tr>
<tr>
<td>Parabola</td>
<td>( = 0 )</td>
<td>( = 1 )</td>
</tr>
<tr>
<td>Ellipse</td>
<td>( &gt; V_{min} ), ( &lt; 0 )</td>
<td>( &gt; 0, &lt; 1 )</td>
</tr>
<tr>
<td>Circle</td>
<td>( V_{min} )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
Many-Particle Systems

- For any many-particle system, there is a “special place” – the center of mass, defined as:

\[ r_{cm} = \frac{\sum m_\alpha r_\alpha}{M} \]

- Newton’s Laws applied to the system as a whole are:

\[ \mathbf{F}_{ext} = M\mathbf{a}_{CM} = \dot{\mathbf{P}} \]
\[ \mathbf{N}_{ext} = \dot{\mathbf{L}} \]

- The angular momentum of the system is given by:

\[ \mathbf{L} = r_{CM} \times \mathbf{P} + \sum_{\alpha} r_\alpha \times p_\alpha \]
Similarly, the kinetic energy of the system can be written as:

\[ T = \frac{1}{2} MV^2 + \frac{1}{2} \sum_\alpha m_\alpha v_\alpha^2 \]

The potential energy includes both external and internal terms

- If the system is in a constant gravitational field,
  \[ U = Mg y_{CM} \]

If only conservative forces act on the system, energy is conserved
Collisions

• Collisions are an interaction between two particles (or systems) that occurs over a finite time interval
• Are analyzed using conservation of momentum and energy
  – Since we don’t usually know the forces acting during the collision
• Momentum is conserved in all collisions…
• …but kinetic energy is not
• If kinetic energy is conserved, the collision is called \textit{elastic}, otherwise it’s \textit{inelastic}
  – Can quantify “degree of inelasticity” of a head-on collision using \textit{coefficient of restitution}:

\[
\varepsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}
\]
• Collisions are most easily analyzed in the CM frame (origin at center of mass of colliding systems) or LAB frame (one of the systems initially at rest)
• If initial impact parameter is unknown (as is the case for collisions between subatomic particles), the results are analyzed using the differential cross section:

\[ \sigma(\theta) = \left. \frac{b}{\sin \theta} \right| \frac{db}{d\theta} \]

• For \( k/r^2 \) type forces,

\[ \sigma(\theta) = \frac{k^2}{(4T_o')^2} \cdot \frac{1}{\sin^4(\theta/2)} \]

Rutherford scattering formula
Rigid Body Motion

• The last major topic of the semester is the motion of rigid systems
  – That is, systems in which the distance between any two particles is constant

• The *inertia tensor* is very useful in describing these systems:

\[
I_{ij} = \sum_{\alpha} m_\alpha \left( \delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)
\]

• With these tensor elements, the rotational energy and angular momentum become:

\[
T_{\text{rot}} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j
\]

\[
L_i = \sum_j I_{ij} \omega_j
\]
• These equations are simplified if we rotate the coordinate system to a new set of axes where the inertia tensor is *diagonal*:

\[
\mathbf{I} = \begin{bmatrix}
I_1 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_3
\end{bmatrix}
\]

  – Since the inertia tensor is both real and symmetric, we can always do this

• To find the principal moments for a given tensor, we must solve the equation:

\[
\begin{vmatrix}
I_{11} - I & I_{12} & I_{13} \\
I_{21} & I_{22} - I & I_{23} \\
I_{31} & I_{32} & I_{33} - I
\end{vmatrix} = 0
\]
• Once the principal moments are found, one can find the principal axes by:
  – Taking the $i$th principal moment and solving the system of equations:

$$
(I_{11} - I_i) \omega_1 + I_{12} \omega_2 + I_{13} \omega_3 = 0
$$

$$
I_{21} \omega_1 + (I_{22} - I_i) \omega_2 + I_{23} \omega_3 = 0
$$

$$
I_{31} \omega_1 + I_{32} \omega_2 + (I_{33} - I_i) \omega_3 = 0
$$

to determine the ratio

$$
\omega_1 : \omega_2 : \omega_3
$$

• This ratio gives the direction of the $i$th principal axis in the original coordinate system
Eulerian Angles

• It’s convenient to always work in the *body frame* where the axes are principal axes
  – Unfortunately, in general that’s *not* an inertial frame
• The rotation that transforms an inertial frame into the body frame is described by the *Eulerian angles* $\theta$, $\phi$, and $\psi$: 
Euler’s Equations for Rigid-Body Motion

• In the body frame, the equations of motion for a rigid body are:

\[ I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = N_1 \]
\[ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_2 \omega_3 = N_2 \]
\[ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_2 \omega_3 = N_3 \]