Lecture 12: More on Generalized Coordinates

- Consider a system of $n$ point particles
  - In rectangular coordinates, $3n$ numbers are needed to specify the positions of all the particles
  - But there may also be $m$ equations of constraint, leaving $3n-m$ degrees of freedom. We abbreviate this number as $s$.

- Therefore, we need $s$ generalized coordinates in the Lagrangian

- In general, the Cartesian coordinates will be functions of the generalized coordinates, and also possibly time:

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t)$$

- $\alpha$ is the particle number (1 to $n$)
- $i$ is the axis (1, 2, or 3)
- $j$ is the generalized coordinate number (1 to $s$)
• We can also define *generalized velocities*:

\[
\dot{q}_j = \frac{dq_j}{dt}
\]

– Note that these do *not* have to have the usual units of velocity (distance/time) – they may be angular velocities or something else entirely

• Again, we can relate the generalized velocities to the “real” velocity for each particle:

\[
\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(\dot{q}_j, t)
\]

• This means that \(T, U,\) and \(L\) can be expressed in terms of the generalized coordinates and velocities:

\[
U = U(q_j, t); \quad T = T(\dot{q}_j, t); \quad L = L(q_j, \dot{q}_j, t)
\]
Hamilton’s Principle Revisited

• Since the kinetic and potential energies are scalars, Hamilton’s Principle is true in configuration space (the space where the axes are the generalized coordinates) as well as “real” space:

When an object moves through configuration space in a given time interval, it will choose the path that minimizes the integral of the difference between the kinetic and potential energies

• In configuration space, we must minimize the following integral:

\[
J = \int_{t_1}^{t_2} L\{\dot{q}_j, q_j; t\} dt
\]
• So Lagrange’s Equations of Motion become:

\[ \frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = 0 \]

• There are \( s \) such equations, one for every degree of freedom in the system
Example: Cycloidal Pendulum

- On one of the early homework assignments, you were asked to determine the equation of motion for the following device:

- This was a pain using Newton’s Laws!
- Now let’s try it using Hamilton’s Principle
- We were told that the path of the bob could be parametrized as:

\[ x = a (\phi - \sin \phi) \]
\[ y = a (\cos \phi - 1) \]
• This means we can take $\phi$ as our generalized coordinate

• The kinetic energy is:

$$T = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right)$$

$$= \frac{1}{2} m \left[ \left( a \left( \dot{\phi} - \dot{\phi} \cos \phi \right) \right)^2 + \left( a \left( -\dot{\phi} \sin \phi \right) \right)^2 \right]$$

$$= \frac{1}{2} m a^2 \dot{\phi}^2 \left[ 1 - 2 \cos \phi + \cos^2 \phi + \sin^2 \phi \right]$$

$$= m a^2 \dot{\phi}^2 \left[ 1 - \cos \phi \right]$$

• And for the potential energy we have:

$$U = mgy = mga \left( \cos \phi - 1 \right)$$
• So we arrive at the Lagrangian:

\[ L = T - U = ma^2 \dot{\phi}^2 [1 - \cos \phi] - mga (\cos \phi - 1) \]

• Now we can determine the equation of motion:

\[
\frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = ma^2 \dot{\phi}^2 \sin \phi + mga \sin \phi
\]

\[- \frac{d}{dt} \left( 2ma^2 \dot{\phi} [1 - \cos \phi] \right)
= ma^2 \dot{\phi}^2 \sin \phi + mga \sin \phi
- 2ma^2 \ddot{\phi} [1 - \cos \phi] - 2ma^2 \dot{\phi}^2 \sin \phi
= mga \sin \phi - 2ma^2 \ddot{\phi} [1 - \cos \phi] - ma^2 \dot{\phi}^2 \sin \phi = 0\]
\[ g \sin \phi - 2a \ddot{\phi} [1 - \cos \phi] - a \dot{\phi}^2 \sin \phi = 0 \]

\[ -\ddot{\phi} = \frac{a \dot{\phi}^2 \sin \phi - g \sin \phi}{2a [1 - \cos \phi]} \]

- It’s not so easy to interpret this yet, so let’s apply some trig identities:

\[ -\ddot{\phi} = \frac{a \dot{\phi}^2 - g}{2a} \sqrt{1 - \cos^2 \phi} \]

\[ = \frac{a \dot{\phi}^2 - g}{2a} \sqrt{(1 - \cos \phi)(1 + \cos \phi)} \]

\[ = \frac{a \dot{\phi}^2 - g}{2a} \frac{\sqrt{1 + \cos \phi}}{\sqrt{1 - \cos \phi}} = \frac{a \dot{\phi}^2 - g}{2a} \frac{\cos \frac{\phi}{2}}{\sin \frac{\phi}{2}} \]
\[-2a\ddot{\phi}\sin\frac{\phi}{2} = \left(a\dot{\phi}^2 - g\right)\cos\frac{\phi}{2}\]

\[-2a\ddot{\phi}\sin\frac{\phi}{2} - a\dot{\phi}^2 \cos\frac{\phi}{2} = -g \cos\frac{\phi}{2}\]

\[4a \frac{d^2}{dt^2}\left(\cos\frac{\phi}{2}\right) = -g \cos\frac{\phi}{2}\]

Simple harmonic motion!

- Though this problem still isn’t trivial (due to the trig identities needed at the end), it’s much easier the Lagrangian way
Forces in the Lagrangian Approach

- Hamilton’s Principle, in which only energy is mentioned, is quite different from Newton’s Laws, which depend on the concept of force
  - …and yet we seem to get to the same result no matter which method is used

- Is that coincidence? No!

- To see why not, start with Lagrange’s Equation in rectangular coordinates:

\[
\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0
\]

\[
\frac{\partial (T - U)}{\partial x_i} - \frac{d}{dt} \frac{\partial (T - U)}{\partial \dot{x}_i} = 0
\]
• In rectangular coordinates, $T$ depends only on $\dot{x}_i$ and $U$ depends only on $x_i$, so we have:

$$\frac{-\partial U}{\partial x_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i} = 0$$

$$F_i = \frac{d}{dt} \left( \frac{\partial}{\partial \dot{x}_i} \left[ \frac{1}{2} m \sum_{j=1}^{3} \dot{x}_j^2 \right] \right) = \frac{d}{dt} [m \ddot{x}_i]$$

$$F_i = m \ddot{x}_i$$

• So the physics in Newton’s Laws and in the Lagrangian method are identical
  – But the ease with which problems can be solved is not!
Generalized Momenta

• In regular Cartesian coordinates, the Lagrangian for a single particle is:

\[ L = T - U = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 - U(x_i) \]

• Given this, we can readily interpret the physical significance of the quantity \( \frac{\partial L}{\partial \dot{x}_i} \):

\[ \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i = p_i \]  It’s the momentum!

• Since there’s nothing special about rectangular Cartesian coordinates in Lagrangian mechanics, we can define a generalized momentum associated with any generalized coordinate:

\[ p_q \equiv \frac{\partial L}{\partial \dot{q}_i} \]
• Keep in mind that, just as a generalized coordinate doesn’t have to have dimensions of length, a generalized momentum doesn’t have to have the usual units for momentum
  – If the generalized coordinate corresponds to an angle, for example, the generalized momentum associated with it will be an angular momentum
• With this definition of generalized momentum, Lagrange’s Equation of Motion can be written as:

\[
\frac{\partial L}{\partial q_j} - \frac{d}{dt} p_j = 0
\]

\[
\dot{p}_j = \frac{\partial L}{\partial q_j}
\]

Just like Newton’s Laws, if we call \( \frac{\partial L}{\partial q_j} \) a “generalized force”