Lecture 18: More on Central-Force Motion

- Last time we wrote down the Lagrangian for a two-particle system, with a central force acting between the particles:

\[ L = \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 - U(r) \]

- But simply by putting the origin at the center of mass of the system, the Lagrangian becomes much simpler:

\[ L = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \right) |\ddot{r}_c|^2 + \frac{1}{2} m_2 \left( \frac{m_1}{m_1 + m_2} \right) |\ddot{r}_c|^2 - U(r) \]

  - Only 3 generalized coordinates, not 6!
The fact that we can write the Lagrangian in this form means that we can treat the system as though it consisted of a single particle of mass $\mu$

- If we want to find the motion of the individual particles, we can easily find $r_1$ and $r_2$ once we know $r$
Reduced Mass: The Extreme Cases

• To get a feel for what the reduced mass represents, let’s consider two extreme cases

• Case 1: \( m_1 \gg m_2 \)

\[
\mu = \frac{m_1 m_2}{m_1 + m_2} \approx \frac{m_1 m_2}{m_1} = m_2
\]

  – Makes sense: it’s a good approximation to treat the heavier particle as stationary

• Case 2: \( m_1 = m_2 \)

\[
\mu = \frac{m_1 m_2}{m_1 + m_2} \approx \frac{m_1^2}{2m_1} = \frac{1}{2} m_1 = \frac{1}{2} m_2
\]

• Note that the reduced mass is always less than the mass of either particle in the system
  – Explains why it’s called “reduced”!
Conserved Quantities

• We can learn quite a bit about the motion of the two-body system simply by considering the conservation laws we’ve already studied
  – even if we don’t know the form of the force acting between the particles
• First, note that with the origin at the center of mass, there is no torque acting anywhere on the system
  – so the angular momentum, $L$, must be constant
• But recall that $L$ is defined as $\mathbf{r} \times \mathbf{p}$
• So for $L$ to be constant, $\mathbf{r}$ and $\mathbf{p}$ must always lie in a given plane
  – Therefore we can reduce the three-dimension problem to two dimensions – we’ve eliminated a degree of freedom
• This means we need just two generalized coordinates to describe the system
  – A convenient choice is the set \((r, \theta)\)
• With these coordinates, the Lagrangian becomes:

\[
L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)
\]

• Note that the variable \(\theta\) doesn’t appear anywhere in the Lagrangian
• This means that the generalized momentum associated with \(\theta\) must be constant:

\[
\dot{p}_\theta = \frac{\partial L}{\partial \dot{\theta}} = 0
\]

\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const} = l
\]
Once again, this allows us to learn something interesting about central forces.

Consider a particle moving past some origin:

\[ dA = \frac{1}{2} r (r\theta) = \frac{1}{2} r^2 d\theta \]

\[ \frac{dA}{dt} = \frac{1}{2} \left( r^2 \dot{\theta} + 2r\dot{r}d\theta \right) \]

This term is 0, since \( d\theta \) is infinitely small.

So the “areal velocity” is just:

\[ \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \]
• But this is closely related to the conserved quantity $l$:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2\mu}$$

• In other words the areal velocity is constant

• You may recall this as Kepler’s Second Law of planetary motion
  
  – He noted this as an experimental fact decades before Newton developed his laws of motion

• We now see that Kepler’s Second Law holds for any central force, not just the $1/r^2$ form of gravity
Energy

• Energy is also conserved for the types of forces we’re considering here:

\[ E = T + U = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r) \]

\[ = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{(\mu r^2 \dot{\theta})^2}{\mu r^2} + U(r) \]

\[ = \frac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \]

• Note that the conservation of angular momentum allows us to write the energy in a form that depends on only one variable
Equations of Central-Force Motion

• This expression for the energy also gives us a first-order differential equation for the motion:

\[ E = \frac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2 \mu r^2} + U(r) \]

\[ \frac{1}{2} \mu \dot{r}^2 = E - U(r) - \frac{l^2}{2 \mu r^2} \]

\[ \dot{r} = \pm \sqrt{\frac{2}{\mu} \left( E - U(r) \right) - \frac{l^2}{\mu^2 r^2}} \]

• Not only is this equation first-order, it is also separable, so we can find the solution by integrating
  – Though, depending on the form of \( U(r) \), the integral might be pretty tough!
• However, in many cases we’re not interested in knowing how $r$ varies with time as much as we’re interested in knowing how it varies with $\theta$
  - i.e., we want to know the path the object takes as it moves around the force center
• To get there, we start with the chain rule:
  \[
  d\theta = \frac{d\theta}{dt} \frac{dt}{dr} dr = \frac{\dot{\theta}}{\dot{r}} dr
  \]
• We know that $\dot{\theta} = l/\mu r^2$ and we have the expression for $\dot{r}$ from the last slide, so:
  \[
  d\theta = \pm \frac{l}{\mu r^2 \sqrt{\frac{2}{\mu} (E - U(r)) - \frac{l^2}{\mu^2 r^2}}} dr = \pm \frac{l/ r^2}{\sqrt{2\mu (E - U(r)) - \frac{l^2}{r^2}}} dr
  \]
• We can now integrate to find $\theta$ as a function of $r$:

$$\theta (r) = \int \frac{\pm l / r^2}{\sqrt{2\mu (E - U(r)) - \frac{l^2}{r^2}}} \, dr$$

• This is valid for any form of $U$, but for most cases we’ll need a computer to evaluate the integral.

• The exceptions are cases where the potential has the form

$$U (r) = kr^n$$

• For some values of $n$, we can express the result in terms of elliptic integrals (see Appendix B of the text)
  – It works out even better if $n = 2$, -1, or -2. We’ll get back to that later.
• We can also find the equation of motion using the Lagrangian approach:

\[
L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U (r)
\]

\[
\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \mu r \ddot{\theta} - \frac{\partial U}{\partial r} - \mu \ddot{r} = 0
\]

\[
\mu (\ddot{r} - r \dot{\theta}^2) = -\frac{\partial U}{\partial r} = F (r)
\]

• Note that this gives a relationship between the motion and the force

• We can better interpret this result by making a change of variables:

\[
u = \frac{1}{r}
\]
• Computing the derivatives of $u$, we find:

$$
\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} = -\frac{1}{r^2} \frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{l} \dot{r}
$$

$$
\frac{d^2 u}{d\theta^2} = \frac{d}{d\theta} \left( -\frac{\mu}{l} \dot{r} \right) = \frac{dt}{d\theta} \frac{d}{dt} \left( -\frac{\mu}{l} \dot{r} \right) = -\frac{\mu}{l} \ddot{r}
$$

• Using the expression for $l$ one more time gives:

$$
\frac{d^2 u}{d\theta^2} = -\frac{\mu^2}{l^2} r^2 \ddot{r}
$$

• With this, we can now find $r\dot{\theta}^2$ and $\ddot{r}$:

$$
r\dot{\theta}^2 = r \left( \frac{l}{\mu r^2} \right)^2 = \frac{l^2}{\mu^2 r^3} = \frac{l^2 u^3}{\mu^2}
$$

$$
\ddot{r} = -\frac{l \dot{\theta}}{\mu} \frac{d^2 u}{d\theta^2} = -\frac{l^2}{\mu^2 r^2} \frac{d^2 u}{d\theta^2} = -\frac{l^2 u^2}{\mu^2} \frac{d^2 u}{d\theta^2}
$$
• We’re now all set to substitute this back into the equation of motion:

\[ \mu (\ddot{r} - r^2 \dot{\theta}) = F(r) \]

\[ \mu \left( -\frac{l^2 u^2}{\mu^2} \frac{d^2 u}{d \theta^2} - \frac{l^2 u^3}{\mu^2} \right) = F \left( \frac{1}{u} \right) \]

\[ \frac{d^2 u}{d \theta^2} + u = -\frac{\mu}{l^2} \frac{1}{u^2} F \left( \frac{1}{u} \right) \]

Putting this in terms of \( r \) again gives:

\[ \frac{d^2}{d \theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r) \]

• If we know \( r(\theta) \), this expression allows us to find the force that must be causing the motion