Lecture 21: Tidal Forces

- For most of the problems we’ll treat this semester, we’ll assume that the Earth is an inertial frame of reference.
- But there are effects that arise precisely because that isn’t exactly true.
- Ocean tides are one example, and since they’re caused by gravity, we’ll look at them now.
- We take a simple model, where the universe consists only of the Earth and the moon, and the Earth is covered entirely by water.
The net gravitational force on the element of water \( m \) is the sum of the forces due to the Earth and the Moon:

\[
F_m = -\frac{G m M_E}{r^2} \mathbf{e}_r - \frac{G m M_m}{R^2} \mathbf{e}_R = m \ddot{r}_m
\]

In principal, we could solve this equation of motion and determine exactly how this element of water moves.

But what we really want to know is how someone on Earth will observe it to move:

– That is, we want the motion relative to the Earth

The first step is writing the equation of motion for the Earth:

\[
M_E \ddot{r}_E = -\frac{G M_m M_E}{D^2} \mathbf{e}_D
\]

Note: both \( r_E \) and \( r_m \) are measured in some inertial frame.
• Therefore the relative motion is given by:

\[ \ddot{r} = \ddot{r}_m - \ddot{r}_E \]
\[ = -\frac{GM_E}{r^2} \mathbf{e}_r - \frac{GM_m}{R^2} \mathbf{e}_R + \frac{GM_m}{D^2} \mathbf{e}_D \]

• The first term is just the usual acceleration towards the center of the Earth (the one that’s equal to \( g \) near the Earth’s surface)

• The other terms only exist due to the presence of the Moon

• We identify these terms as the tidal acceleration:

\[ \mathbf{a}_T = -GM_m \left( \frac{\mathbf{e}_R}{R^2} - \frac{\mathbf{e}_D}{D^2} \right) \]
Magnitude of the Tidal Force

- We can now find the tidal force at any point $P$ on the Earth’s surface:

$$
sin \xi = \frac{y}{R} \quad \text{and} \quad \cos \xi = \frac{D + x}{\sqrt{y^2 + (D + x)^2}} = \frac{D + x}{R}.
$$

where $e_D = -1i$, $e_R = -\cos \xi i - \sin \xi j$. 

To Moon

Earth’s rotation

\[ \xi \]

\[ P \]
Therefore, the $x$ component of the tidal acceleration is:

$$a_{T,x} = -GM_m \left( \frac{D + x}{R^3} - \frac{1}{D^2} \right)$$

$$= -\frac{GM_m}{D^2} \left( \frac{D + x}{\left( R \right)^2} \right) \left( R - 1 \right) = -\frac{GM_m}{D^2} \left( \frac{1 + x / D}{\left( R / D \right)^3} - 1 \right)$$

$$= -\frac{GM_m}{D^2} \left( \frac{1 + x / D}{\left[ \left( \frac{y}{D} \right)^2 + \left( \frac{D + x}{D} \right)^2 \right]^{3/2}} - 1 \right)$$

$$= -\frac{GM_m}{D^2} \left( \frac{1 + x / D}{\left[ 1 + \frac{2x}{D} + \left( \frac{x}{D} \right)^2 + \left( \frac{y}{D} \right)^2 \right]^{3/2}} - 1 \right)$$
• So far we’ve kept the exact solution, but now we make use of the fact that the radius of the Earth is much less than $D$ to get an approximate answer:

$$a_{T,x} \approx \frac{-GM_m}{D^2} \left( \frac{1 + x/D}{1 + \frac{2x}{D}} \right)^{3/2} - 1$$

$$\approx \frac{-GM_m}{D^2} \left[ (1 + x/D) \left(1 - \frac{3x}{D}\right) - 1 \right]$$

$$= \frac{-GM_m}{D^2} \left[ 1 - \frac{3x}{D} + \frac{x}{D} - 1 \right] = \frac{2GM_m x}{D^3}$$

• Plugging through the same math for the $y$ component yields:

$$a_{T,y} = -\frac{GM_m y}{D^3}$$
Notes on Tidal Acceleration

• Note that the $x$ component always points in the same direction as the displacement

• This is why there are two “high tides” per day from the moon (one corresponds to $x = +R_E$, and the other to $x = -R_E$)

• The Sun also produces tides on the Earth
  – We might expect this to be the dominant contribution, since the Sun’s mass is so much greater than the Moon’s
  – But, tides are due to the change in the gravitational force across the diameter of the earth
  – It turns out that’s the sun’s contribution to tides is about half as big as the moon’s
Central-Force Motion Using Lagrangians

• We can also find the equation of motion using the Lagrangian approach:

\[ L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \]

\[ \frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \mu r \dot{\theta}^2 - \frac{\partial U}{\partial r} - \mu \ddot{r} = 0 \]

\[ \mu (\ddot{r} - r \dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \]

• Note that this gives a relationship between the motion and the force

• We can better interpret this result by making a change of variables:

\[ u = \frac{1}{r} \]
• Computing the derivatives of $u$, we find:

$$
\frac{d}{d \theta} u = -\frac{1}{r^2} \frac{d}{d \theta} r = -\frac{1}{r^2} \frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{l} \dot{r}
$$

$$
\frac{d^2}{d \theta^2} u = \frac{d}{d \theta} \left(-\frac{\mu}{l} \dot{r}\right) = \frac{d}{d \theta} \frac{d}{d t} \left(-\frac{\mu}{l} \dot{r}\right) = -\frac{\mu}{l \dot{\theta}} \ddot{r}
$$

• Using the expression for $l$ one more time gives:

$$
\frac{d^2}{d \theta^2} u = -\frac{\mu^2}{l^2} r^2 \ddot{r}
$$

• With this, we can now find $r \dot{\theta}^2$ and $\ddot{r}$:

$$
r \dot{\theta}^2 = r \left(\frac{l}{\mu r^2}\right)^2 = \frac{l^2}{\mu^2 r^3} = \frac{l^2 u^3}{\mu^2}
$$

$$
\ddot{r} = -\frac{l^2}{\mu^2 r^2} \frac{d^2}{d \theta^2} u = -\frac{l^2 u^2}{\mu^2} \frac{d^2}{d \theta^2} u
$$
• We’re now all set to substitute this back into the equation of motion:

\[ \mu (\ddot{r} - r^2 \dot{\theta}) = F(r) \]

\[ \mu \left( -\frac{l^2 u^2}{\mu^2} \frac{d^2 u}{d \theta^2} - \frac{l^2 u^3}{\mu^2} \right) = F\left( \frac{1}{u} \right) \]

\[ \frac{d^2 u}{d \theta^2} + u = -\mu \frac{1}{l^2 u^2} F\left( \frac{1}{u} \right) \]

Putting this in terms of \( r \) again gives:

\[ \frac{d^2}{d \theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r) \]

• If we know \( r(\theta) \), this expression allows us to find the force that must be causing the motion