Lecture 24: Orbital Dynamics, and Introduction to Many-Particle Systems

• We now consider what is involved in changing a satellite’s orbit
  – For example, a mission to Mars requires taking a spacecraft from Earth’s orbit and placing it in Mars’ orbit
• For simplicity, we’ll assume that both the Earth and Mars have circular orbits
  – Not a terrible approximation: $\varepsilon = 0.02$ for the Earth and 0.09 for Mars
• It turns out the the most energy-efficient way of doing this is known as a Hohmann transfer
Hohmann Transfer Diagram

- Initial orbit
- Final orbit
- Transfer orbit
- First rocket burn
- Second rocket burn

$r_1$ and $r_2$ represent the radii of the initial and transfer orbits, respectively.
Hohmann Transfer

- This transfer consists of two velocity changes (think of them as two fast rocket burns), both tangential to the velocity:
  1. The first burn changes $E$ and $l$, turning the initial circular orbit into an elliptical orbit with apocenter equal to the final orbit’s radius
     - This elliptical orbit is called the transfer orbit
  2. When the spacecraft reaches the apocenter of the transfer orbit, a second burn changes the orbit into a circular one
- The energy and velocity of the initial orbit are:

$$E = -\frac{k}{2r_1} = T + U = \frac{1}{2}mv_1^2 - \frac{k}{r_1}$$

$$v_1 = \sqrt{\frac{k}{mr_1}}$$
• For the transfer orbit, the major axis must be the sum of the radii of the initial and final circular orbits:

\[ 2a_t = r_1 + r_2 \]

• Therefore the energy of the transfer orbit must be:

\[ E = -\frac{k}{2a_t} = -\frac{k}{r_1 + r_2} \]

• With this, we can find the velocity just after the first rocket burn:

\[
E = -\frac{k}{r_1 + r_2} = T + U = \frac{1}{2}m v_{t1}^2 - \frac{k}{r_1}
\]

\[
v_{t1}^2 = \frac{2}{m} \left[ \frac{k}{r_1} - \frac{k}{r_1 + r_2} \right] = \frac{2k}{m} \left[ \frac{r_2}{r_1(r_1 + r_2)} \right]
\]
The initial velocity change needed is
\[ \Delta v_1 = v_{1t} - v_1 \]

We can also find the velocity of the transfer orbit when it reaches \( r_2 \):

\[
\begin{align*}
- \frac{k}{r_1 + r_2} &= \frac{1}{2} mv_{t2}^2 - \frac{k}{r_2} \\
\Rightarrow v_{t2} &= \sqrt{\frac{2k}{m} \left[ \frac{k}{r_2} - \frac{k}{r_1 + r_2} \right]} = \sqrt{\frac{2k}{m} \left[ \frac{r_1}{r_2 (r_1 + r_2)} \right]} \\
\end{align*}
\]

While we know a circular orbit at \( r_2 \) must have velocity
\[ v_2 = \sqrt{\frac{k}{mr_2}} \]
• Which means that the second rocket burn must add a velocity:

\[ \Delta v_2 = v_2 - v_{2t} \]

• The time the transfer will take is given by half of the period of the transfer orbit. For Kepler’s Third Law, this is:

\[ T_t = \pi \sqrt{\frac{m}{k} a_t^{3/2}} = \pi \sqrt{\frac{1}{GM_{\text{sun}}} a_t^{3/2}} \]

• From here to Mars, that’s:

\[ T_t = \pi \sqrt{\frac{1}{(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)(2 \times 10^{30} \text{ kg}) (1.9 \times 10^{11} \text{ m})^{3/2}}} \]

\[ = 2.2 \times 10^7 \text{ s} = 0.7 \text{ yr} \]

• It would take over 44 years to get to Pluto this way!
Dynamics of Systems of Particles

• So far we’ve dealt with the motion of single particles
  – Even for central-force motion between two particles we used a trick to treat it as single-particle motion
• We now want to extend our understanding of mechanics to situations where the single-particle treatment is not applicable
• One such area is rotation, a concept that doesn’t even exist for a point particle
• Another is collisions, in which two or more particles interact
• Our immediate goal is to develop the tools that we’ll need to handle these multi-particle systems
Newton’s Third Law, Revisited

- Newton’s Third Law will prove very useful in dealing with multi-particle systems.
- There are actually two forms of this Law:
  1. “Weak form”: The forces exerted by two particles on each other are equal in magnitude and opposite in direction:
      \[ f_{12} = -f_{21} \]
  2. “Strong form”: Not only does the weak form hold, but the forces are also directed along the line between the particles.
- We’re going to assume the strong form holds, but remember that there are exceptions.
  - For example, magnetic forces obey the weak form, but not the strong form, of the Third Law.
Center of Mass

- We’ve already seen the concept of center of mass for two-body systems in a few problems. If the origin was at the center of mass, we found that:

\[ m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0 \]

- If the origin was anywhere else, this would become:

\[ \mathbf{r}_{cm} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \]

- Extending this to a system of \( N \) point particles gives:

\[ \mathbf{r}_{cm} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \]

- For a continuous distribution of matter the sum becomes an integral:

\[ \mathbf{r}_{cm} = \frac{1}{M} \int \mathbf{r} dm \]
Change in Linear Momentum of the System

- The linear momentum of a system of particles is simply the vector sum of the momentum of each particle:

\[ \mathbf{p} = \sum_{\alpha=1}^{N} \mathbf{p}_\alpha \]

so the rate of change of the momentum is:

\[ \dot{\mathbf{p}} = \sum_{\alpha=1}^{N} \dot{\mathbf{p}}_\alpha = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha \]

- The force acting on each particle can be broken into two components, according to the cause of the force:
  1. External forces are caused by things outside of the system of particles we’re considering. We call these \( \mathbf{F}^e \)
  2. Internal forces are caused by the other particles in the system. We can write these as:

\[ \mathbf{f}_\alpha = \sum_{\beta} \mathbf{f}_{\alpha\beta} \]
• So the change in momentum is:

\[ \sum_{\alpha=1}^{N} \dot{p}_\alpha = \sum_{\alpha=1}^{N} \left( F^e_\alpha + \sum_{\beta=1}^{N} f_{\alpha\beta} \right) = \sum_{\alpha=1}^{N} F^e_\alpha + \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} f_{\alpha\beta} \]

• The first term on the right-hand side is the total external force on the system, which we’ll call \( F \)

• The second term can be written as:

\[ \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} f_{\alpha\beta} = \sum_{\alpha>\beta} f_{\alpha\beta} + \sum_{\alpha<\beta} f_{\alpha\beta} = \sum_{\alpha>\beta} f_{\alpha\beta} + \sum_{\beta<\alpha} f_{\beta\alpha} \]

\[ = \sum_{\alpha>\beta} (f_{\alpha\beta} + f_{\beta\alpha}) = \sum_{\alpha>\beta} (f_{\alpha\beta} - f_{\alpha\beta}) = 0 \]
• Thus we have:

\[ \mathbf{p} = \sum_{\alpha=1}^{N} \mathbf{p}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha = M \mathbf{r}_{CM} \]

\[ \dot{\mathbf{p}} = M \dot{\mathbf{r}}_{CM} = \mathbf{F} \]

• So the linear momentum of the entire system behaves as it would for a single particle of mass \( M \) located at the center of mass

• Internal forces can never change the linear momentum of a system
  – Note that we only used the weak form of Newton’s Third Law in our derivation, so this is true even for systems with magnetic forces