Lecture 27: Elastic Collisions

- Last time, we found that the following relations were true for any elastic collision (as viewed in the CM frame):
  - Conservation of momentum:
    
    \[ m_1 u_1' = m_2 u_2' \]
    
    \[ m_1 v_1' \cos \theta = m_2 v_2' \cos \theta \]
    
    \[ m_1 v_1' \sin \theta = m_2 v_2' \sin \theta \]

  - Conservation of kinetic energy:
    
    \[ \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \]
Using the momentum conservation to write all the energies in terms of $u'_1$ and $v'_1$ we find:

$$m_1 u'_1^2 + m_2 \left( \frac{-m_1 u'_1}{m_2} \right)^2 = m_1 v'_1^2 + m_2 \left( \frac{-m_1 v'_1}{m_2} \right)^2$$

$$u'_1^2 \left( 1 + \frac{m_1}{m_2} \right) = v'_1^2 \left( 1 + \frac{m_1}{m_2} \right)$$

$$u'_1 = v'_1$$

Similarly, we find that $u'_2 = v'_2$

In the CM frame, each particle exits from the collision with the same speed that it entered – only the direction of the velocity may change
• To see how the collision looks in the LAB frame, we transform all of the velocity vectors from the LAB frame to the CM frame

• In the LAB frame, the center of mass position is given by:

\[
\frac{m_1 r_1 + m_2 r_2}{M} = R_{\text{CM}}
\]

which means the velocity of the center of mass is:

\[
\frac{m_1 u_1 + m_2 u_2}{M} = V_{\text{CM}}
\]

• But recall that \(m_2\) is at rest in the LAB frame, so \(u_2 = 0\). Therefore,

\[
\frac{m_1 u_1}{M} = \frac{m_1 u_1}{m_1 + m_2} = V_{\text{CM}}
\]

The CM frame is moving towards \(m_2\) with speed

\[
\frac{m_1 u_1}{m_1 + m_2} = V_{\text{CM}}
\]
• The relative initial velocities are the same in the LAB and CM frames, and equal to \( u_1 \)

\[
u_1 = u'_1 + u'_2
\]

• This fact that \( m_2 \) is at rest in the LAB frame means that its speed in the CM frame must be \( V_{CM} \):

\[
u'_2 = V_{CM} = \frac{m_1u_1}{m_1 + m_2}
\]

• We also know that:

\[
v'_2 = u'_2 = \frac{m_1u_1}{m_1 + m_2}
\]

\[
v'_1 = u'_1 = u_1 - u'_2 = u_1 - \frac{m_1u_1}{m_1 + m_2} = u_1 \left(1 - \frac{m_1}{m_1 + m_2}\right)
\]

\[
= \frac{m_2u_1}{m_1 + m_2}
\]
We now focus on the motion of \( m_1 \), using our knowledge of non-relativistic (aka Galilean) velocity transformations:

- Velocity components perpendicular to the boost direction are constant (this is true in relativity, too)
- Velocity components along the boost direction differ by the boost velocity

Applying these facts to the motion of \( m_1 \) after the collision in both the CM and LAB frames, we find:

\[

\begin{align*}
    v'_1 \sin \theta &= v_1 \sin \psi \\
    v'_1 \cos \theta + V_{CM} &= v_1 \cos \psi
\end{align*}

\]

Dividing these equations gives:

\[
    \tan \psi = \frac{v'_1 \sin \theta}{v'_1 \cos \theta + V_{CM}} = \frac{\sin \theta}{\cos \theta + V_{CM} / v'_1}
\]
• The relationship between \( V_{CM} \) and \( v'_1 \) is:

\[
\frac{V_{CM}}{v'_1} = \frac{m_1 u_1}{m_1 + m_2} = \frac{m_1}{m_2}
\]

• Thus we find the relationship between the scattering angle in the LAB frame and that in the CM frame

\[
\tan \psi = \frac{\sin \theta}{\cos \theta + V_{CM} / v'_1} = \frac{\sin \theta}{\cos \theta + m_1 / m_2}
\]
• Plots of $\psi$ vs. $\theta$ for some values of $m_1/m_2$:

$m_1/m_2 = 0$

$m_1/m_2 = 0.5$

$m_1/m_2 = 1.0$

$m_1/m_2 = 1.5$

Note that if $m_1 > m_2$ there are two possible $\theta$ values for every $\psi$. 
Energy in Elastic Collisions

• We can also think about collisions in terms of the energy
• In particular, we’ll explore the final kinetic energy of $m_1$ in the LAB frame
• But to start, let’s write down the initial kinetic energy of the system, in both the LAB and CM frames:

LAB:

$$T_0 = \frac{1}{2} m_1 u_1^2$$

True since $u_2 = 0$

CM:

$$T_0' = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$$

• Using what we know about $u_1'$ and $v_2'$ in terms of $u_1$, we can rewrite this as:
\[
T'_0 = \frac{1}{2} m_1 \left( \frac{m_2 u_1}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left( \frac{m_1 u_1}{m_1 + m_2} \right)^2
\]

\[
= \frac{m_1 m_2 u_1^2}{2 (m_1 + m_2)^2} [m_1 + m_2] = \frac{m_1 m_2 u_1^2}{2 (m_1 + m_2)} = \frac{m_2}{m_1 + m_2} T_0
\]

This shows that the kinetic energy measured in the CM frame is smaller than that measured in any other frame. Consistent with the notion that \(T\) is the sum of energy due to motion within the system plus center-of-mass energy of the system.
• The fraction of the kinetic energy retained by $m_1$ after the collision in the LAB frame is:

$$\frac{T_1}{T_0} = \frac{1}{2} \frac{m_1 v_1^2}{m_1 u_1^2} = \frac{v_1^2}{u_1^2}$$

• To relate this to the LAB scattering angle, we use conservation of energy and momentum:

$$p_x = m_1 u_1 = m_1 v_1 \cos \psi + m_2 v_2 \cos \zeta$$

$$p_y = 0 = m_1 v_1 \sin \psi - m_2 v_2 \sin \zeta$$

$$T = \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$
From the energy equation, we find:

\[ v_2 = \sqrt{\frac{m_1}{m_2} (u_1^2 - v_1^2)} \]

Plugging this into the \( p_y \) equation gives:

\[ m_1 v_1 \sin \psi = m_2 \sqrt{\frac{m_1}{m_2} (u_1^2 - v_1^2)} \sin \zeta \]

\[ \sin \zeta = \frac{m_1 v_1 \sin \psi}{\sqrt{m_1 m_2 (u_1^2 - v_1^2)}} = \sqrt{1 - \cos^2 \zeta} \]

\[ \cos \zeta = \sqrt{1 - \frac{m_1^2 v_1^2 \sin^2 \psi}{m_1 m_2 (u_1^2 - v_1^2)}} = \sqrt{\frac{m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi}{m_2 (u_1^2 - v_1^2)}} \]
• This means that:

\[ m_2 v_2 \cos \zeta = m_2 \sqrt{\frac{m_1 (u_1^2 - v_1^2)}{m_2}} \frac{m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi}{m_2 (u_1^2 - v_1^2)} \]

\[ = \sqrt{m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right]} \]

• We can now insert this into the \( p_x \) equation to find:

\[ m_1 u_1 = m_1 v_1 \cos \psi + \sqrt{m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right]} \]

\[ m_1 (u_1 - v_1 \cos \psi) = \sqrt{m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right]} \]

\[ m_1^2 u_1^2 - 2m_1 v_1 \cos \psi + m_1^2 v_1^2 \cos \psi = m_1 \left[ m_2 (u_1^2 - v_1^2) - m_1 v_1^2 \sin^2 \psi \right] \]
From which we can find \( v_1/u_1 \) with the quadratic formula:

\[
\frac{v_1}{u_1} = \frac{2m_1 \cos \psi \pm \sqrt{4m_1^2 \cos^2 \psi - 4(m_1^2 - m_2^2)}}{2(m_1 + m_2)}
\]

\[
= \frac{m_1}{m_1 + m_2} \left[ \cos \psi \pm \sqrt{\cos^2 \psi - \left(1 - \frac{m_2^2}{m_1^2}\right)} \right]
\]

\[
= \frac{m_1}{m_1 + m_2} \left[ \cos \psi \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2 \psi} \right]
\]
Inelastic Collisions

- All collisions conserve energy (just like elastic collisions do)
- But, if *systems* of particles collide, the internal energy of the systems may change
  - Therefore, the kinetic energy associated with the motion of the center of mass is not conserved
- Such collisions are called *inelastic*
  - The extreme case is a collision between two objects that stick together after they collide (two blobs of clay or silly putty might behave this way)
  - These collisions are called *totally inelastic*
- Most collisions lie somewhere between elastic and totally inelastic
Quantifying Inelasticity

• There are two ways to measure how inelastic a collision is

1. Measure the kinetic energy before and after the collision, and call the difference \( Q \):

\[
T_f - T_i = Q
\]

\[
Q + \frac{1}{2} m_1^2 u_1^2 + \frac{1}{2} m_2^2 u_2^2 = \frac{1}{2} m_1^2 v_1^2 + \frac{1}{2} m_2^2 v_2^2
\]

The larger \(|Q|\) is, the more inelastic the collision

– If \( Q > 0 \), the final kinetic energy is \textit{greater} than the initial. Such collisions are called \textit{exoergic} – an explosion is one example

– If \( Q < 0 \), the final kinetic energy is less than the initial. These collisions are \textit{endoergic} – a collision between blobs of clay falls into this category
Coefficient of Restitution

2. Another way to think about inelasticity is to consider the relative velocity in a head-on collision. In the CM frame:

Elastic collision

\[ \frac{|v_1 - v_2|}{|u_1 - u_2|} = 1 \]

- In general we define \( \frac{|v_1 - v_2|}{|u_1 - u_2|} = \varepsilon \) as the coefficient of restitution.
- For head-on collisions in non-CM reference frames, the velocity components normal to the collision plane enter the formula.

Totally inelastic collision

\[ m_1 + m_2 \quad v_1 = v_2 = 0 \]

\[ \frac{|v_1 - v_2|}{|u_1 - u_2|} = 0 \]
Impulse

- Even though we may not know what forces act during a collision, we can determine something about those forces from Newton’s Second Law.
- During the collision:
  \[ F = \dot{p} = \frac{d(mv)}{dt} \]
  \[ Fdt = d(mv) \]
  \[ \int_{t_1}^{t_2} Fdt = \Delta (mv) = m\Delta v \] if \( m \) is constant

- The quantity \( \Delta(mv) \) is called the \textit{impulse}, and given the symbol \( P \).
- The left-hand side is related to the time-average of the force acting during the collision, so we have:
  \[ F_{\text{avg}} \Delta t = P \]