Lecture 3: Resistive forces, and Energy

• Last time we found the velocity of a projectile moving with air resistance:

\[
\begin{align*}
    v_x(t) &= v_{x,o} e^{-\frac{kt}{m}} \\
    v_y(t) &= -\frac{mg}{k} + \left(v_{y,o} + \frac{mg}{k}\right) e^{-\frac{kt}{m}}
\end{align*}
\]

• One more integration gives us the position as a function of time:

\[
\begin{align*}
    \frac{dx}{dt} &= v_{x,o} e^{-\frac{kt}{m}} \\
    \frac{dy}{dt} &= -\frac{mg}{k} + \left(v_{y,o} + \frac{mg}{k}\right) e^{-\frac{kt}{m}} \\
    x(t) &= -\frac{m}{k} v_{x,o} e^{-\frac{kt}{m}} + C \\
    y(t) &= -\frac{mg}{k} t - \frac{m}{k} \left(v_{y,o} + \frac{mg}{k}\right) e^{-\frac{kt}{m}} + C \\
    x(t) &= x_o + \frac{m}{k} v_{x,o} \left(1 - e^{-\frac{kt}{m}}\right) \\
    y(t) &= y_o - \frac{mg}{k} t + \frac{m}{k} \left(v_{y,o} + \frac{mg}{k}\right) \left(1 - e^{-\frac{kt}{m}}\right)
\end{align*}
\]
• Let’s again find the range of the projectile
• Rather than writing $y$ in terms of $x$, we can solve for the time of flight, and plug that value in to the equation for $x$

\[
y(t_F) = y_o = y_o - \frac{mg}{k} t_F + \frac{m}{k} \left(v_{y,o} + \frac{mg}{k}\right) \left(1 - e^{-\frac{k}{m}t_F}\right)
\]

\[
t_F = \frac{1}{g} \left(v_{y,o} + \frac{mg}{k}\right) \left(1 - e^{-\frac{k}{m}t_F}\right)
\]

• This is a transcendental equation, meaning there is no exact solution
  – However, as physicists, we’re not allowed to give up!
  – We need to find the best approximate solution that we can
• Of course, it’s easy to do on a computer, but if we’re stuck on a deserted island (or doing P321 homework)….
Perturbative Solution (for Small $k/m$)

- If we assume that $k/m$ is small, we can use a Taylor expansion to find an approximate answer for $t_F$:

$$t_F = \frac{1}{g} \left( v_{y,o} + \frac{mg}{k} \right) \left( 1 - \left[ 1 - \frac{k}{m} t_F + \frac{1}{2} \left( \frac{k}{m} t_F \right)^2 - \frac{1}{6} \left( \frac{k}{m} t_F \right)^3 + \ldots \right] \right)$$

$$\approx \frac{1}{g} \left( v_{y,o} + \frac{mg}{k} \right) \left( \frac{k}{m} t_F - \frac{1}{2} \left( \frac{k}{m} t_F \right)^2 + \frac{1}{6} \left( \frac{k}{m} t_F \right)^3 \right)$$

$$\frac{g}{\left( v_{y,o} + \frac{mg}{k} \right)} = \left( \frac{k}{m} - \frac{1}{2} \left( \frac{k}{m} \right)^2 t_F + \frac{1}{6} \left( \frac{k}{m} \right)^3 t_F^2 \right)$$

$$\frac{1}{2} \left( \frac{k}{m} \right) t_F = 1 - \frac{mg}{k \left( v_{y,o} + \frac{mg}{k} \right)} + \frac{1}{6} \left( \frac{k}{m} \right)^2 t_F^2 = \frac{v_{y,o}}{\left( v_{y,o} + \frac{mg}{k} \right)} + \frac{1}{6} \left( \frac{k}{m} \right)^2 t_F^2$$
\[
\frac{1}{2} \left( \frac{k}{m} \right) t_F = 1 - \frac{mg}{k \left( v_{y,o} + \frac{mg}{k} \right)} + \frac{1}{6} \left( \frac{k}{m} \right)^2 t_F^2 = \frac{k v_{y,o}}{(k v_{y,o} + mg)} + \frac{1}{6} \left( \frac{k}{m} \right)^2 t_F^2
\]

\[
t_F = \frac{2 m v_{y,o}}{k v_{y,o} + mg} + \frac{1}{3} \left( \frac{k}{m} \right) t_F^2 = \frac{2 v_{y,o}}{k v_{y,o}} \left( g + mg + \frac{1}{3} \left( \frac{k}{m} \right) \right)
\]

\[
t_F = \frac{2 v_{y,o}}{g} \left( 1 - \frac{k v_{y,o}}{mg} + \ldots \right) + \frac{1}{3} \left( \frac{k}{m} \right) t_F^2
\]

\[
= \frac{2 v_{y,o}}{g} + \frac{k}{m} \left( \frac{t_F^2}{3} - \frac{2 v_{y,o}^2}{g^2} \right)
\]

- The first term has the expansion parameter in the denominator, so it also needs to be expanded:

- Note that if \( k \) is zero, we get exactly the answer we had before considering air resistance (if we didn’t, we’d need to check for errors!)

- The equation for \( t_F \) is quadratic, which we can solve exactly
• However, we can get a good approximation by making the assumption that the solution has the form:

\[ t_F = t_o + t_1 \]

where \( t_1 \) is small

• Plugging this into our equation gives:

\[
t_o + t_1 = t_o + \frac{k}{m} \left( \frac{(t_o + t_1)^2}{3} - \frac{2v_{y,o}^2}{g^2} \right)
\]

\[
\approx t_o + \frac{k}{m} \left( \frac{t_o^2}{3} - \frac{2v_{y,o}^2}{g^2} \right)
\]

\[
= \frac{2v_{y,o}}{g} + \frac{k}{m} \left( \frac{(2v_{y,o} / g)^2}{3} - \frac{2v_{y,o}^2}{g^2} \right)
\]

\[
= \frac{2v_{y,o}}{g} \left( 1 + \frac{2kv_{y,o}}{3mg} - \frac{kv_{y,o}}{mg} \right) = \frac{2v_{y,o}}{g} \left( 1 - \frac{kv_{y,o}}{3mg} \right)
\]
• The last step, where assumed that the solution is only slightly different from an answer we already know, is the distinctive feature of the approximation technique called the *perturbative method*.

• This technique is very important throughout physics
  - You are sure to see it again in Quantum Mechanics
  - In fact, in what we currently believe is the correct description of nature, quantum field theory, we are unable to exactly solve the equations for *anything*
  - Perturbative techniques are our best means of arriving at precise numerical predictions that can be tested in experiments (though some researchers are making great strides in using computers to find approximate solutions)
Work and Energy

- We define the following quantity as work:

\[ W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} \]

This is an example of a path integral

- It seems like an odd definition, but it has one very interesting property:

\[
\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \int_{r_1}^{r_2} m \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} = \int_{t_1}^{t_2} m \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt} \, dt \\
= \int_{t_1}^{t_2} m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \, dt = \int_{t_1}^{t_2} \frac{1}{2} \frac{d}{dt} \left( m\mathbf{v} \cdot \mathbf{v} \right) \, dt \\
= \int_{v_1^2}^{v_2^2} \frac{1}{2} \, m \mathbf{v}^2 \, d\mathbf{v} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2
\]
• If we now define the quantity $\frac{1}{2}mv^2$ as the kinetic energy $(T)$, we have:

$$W = \Delta T$$

– This is the work-energy theorem

• This provides a way to determine the change in velocity of an object without integrating the equations of motion

• In the definition of work, $F$ is the total force acting on the object

• Now consider the case where multiple forces are acting:
  – for example, imagine lifting a box from the floor to a shelf
  – There are two forces at work – gravity and the force supplied by the lifter

• If the box is lifted at constant velocity, there is no change in kinetic energy in the process
• Does this agree with the work-energy theorem?

Yes! The net force acting on the box is 0, so

\[ W = \int \mathbf{F} \cdot d\mathbf{r} = 0 = \Delta T \]

– but it also means the work-energy theorem isn’t very useful in this case

• But can we say anything interesting about the work done by the gravitational force alone?

• Force as a function of position is a vector field
  – for gravity, it’s \( \mathbf{F} = -mg \mathbf{j} \)

• In this case, we can rewrite the force in terms of a scalar field:

\[ \mathbf{F} = -\frac{\partial}{\partial y} (mgy) \mathbf{j} = -\nabla (mgy) \]
• If we call this scalar field $U$, we can express the work done by gravity as:

$$W = \int \mathbf{F} \cdot d\mathbf{r} = -\int \nabla U \cdot d\mathbf{r}$$

$$= -\Delta U$$

• We call $U$ the potential energy associated with the gravitational force
  – Note that it has the same units as kinetic energy

• Note also that the value of the path integral in this case depends only on the endpoints of the path, not on the path itself
  – i.e., all of the following paths have the same $\int \mathbf{F} \cdot d\mathbf{r}$

![Diagram of paths from 1 to 2]
When can we find a potential energy?

• Forces for which the work done between two points is independent of the path taken are called conservative
  – The requirement that the force be of the form $\mathbf{F} = -\nabla U$ is an equivalent definition

Potential energy can only be defined for conservative forces

• But how can we tell whether a given force is conservative?
• Note that:
  \[
  \nabla \times (\nabla U) = \sum_{ijk} \varepsilon_{ijk} \frac{\partial (\nabla U)^k}{\partial x_j} \mathbf{e}_i = \sum_{ijk} \varepsilon_{ijk} \frac{\partial^2 U}{\partial x_j \partial x_k} \mathbf{e}_i
  \]

  Defined as: $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$
  $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$
  $\varepsilon_{ijk} = 0$ if any index is repeated
• Using the definition of $\varepsilon_{ijk}$ we see that

$$\nabla \times (\nabla U) = \sum_{ijk} \left( \frac{\partial^2 U}{\partial x_j \partial x_k} - \frac{\partial^2 U}{\partial x_k \partial x_j} \right) e_i = 0$$

Any force which has a curl equal to zero is conservative.

• To get an intuitive picture, a force with a non-zero curl has lines of force that form closed loops:

As a particle travels around this loop, the work done by $\mathbf{F}$ continues to increase.

Potential energy is not meaningful.
When Is Energy Useful?

• Clearly, if only conservative forces act on a particle, energy can be used to solve problems.
• However, even if there are non-conservative forces, energy is still useful if the non-conservative forces don’t do any work.
• Example: An object sliding down a frictionless ramp of arbitrary shape.

Both gravity and the normal force $F_N$ act on the object.
$F_N$ is not conservative.
But since $F_N \cdot dr = 0$, the normal force does no work.
Can find speed as a function of height using energy.