Lecture 30: Motion in Non-Inertial Frames

• So far we’ve been careful to choose inertial reference frames
  – Those are the ones, after all, where Newton’s Laws apply
• But sometimes it’s easier to analyze a problem in a non-inertial frame
  – We’ll see an example when we study rotational motion
• We then have to explicitly account for the non-inertial motion of the reference frame
  – We’ll find that we need to introduce “fictional” forces if we want to apply Newton’s Laws in these frames
Example: Rotating frame

- Let’s assume we have one inertial reference frame, and a second frame that’s rotating with respect to it:
  - Primed quantities are in inertial frame

\[
dr' = (dr)_{\text{fixed}} = d\theta \times r
\]

In the inertial (or fixed) frame, the motion of \( P \) is given by:

\( P \) is at rest is the rotating frame
• Taking the time derivative of the previous equation, we have:

\[
\left( \frac{dr}{dt} \right)_{\text{fixed}} = \frac{d\theta}{dt} \times r = \omega \times r
\]

• This is true if \( P \) is at rest in the rotating frame. If it’s moving in that frame, we have:

\[
\left( \frac{dr}{dt} \right)_{\text{fixed}} = \left( \frac{dr}{dt} \right)_{\text{rotating}} + \omega \times r
\]

• Although we derived the above equation using the position vector as an example, there’s nothing special about \( r \) – all vectors will behave the same way. So, in general:

\[
\left( \frac{dQ}{dt} \right)_{\text{fixed}} = \left( \frac{dQ}{dt} \right)_{\text{rotating}} + \omega \times Q
\]
• Let’s now see how the velocity measured in the fixed frame is related to that measured in the rotating frame.

• Recall the relation between vectors in the two frames:

\[ \mathbf{r}' = \mathbf{R} + \mathbf{r} \]

• Taking the time derivative, we get:

\[ \left( \frac{d\mathbf{r}'}{dt} \right)_{\text{fixed}} = \left( \frac{d\mathbf{R}}{dt} \right)_{\text{fixed}} + \left( \frac{d\mathbf{r}}{dt} \right)_{\text{fixed}} \]

• Now we put in what we already know about \( \left( \frac{d\mathbf{r}}{dt} \right)_{\text{fixed}} \) to get:

\[ \left( \frac{d\mathbf{r}'}{dt} \right)_{\text{fixed}} = \left( \frac{d\mathbf{R}}{dt} \right)_{\text{fixed}} + \left( \frac{d\mathbf{r}}{dt} \right)_{\text{rotating}} + \mathbf{\omega} \times \mathbf{r} \]
• We can write the previous equation in shorthand:

\[ \mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \mathbf{\omega} \times \mathbf{r} \]

velocity in the inertial frame
velocity in the noninertial frame

• This gives us the formula to translate velocities measured in the two frames

• But acceleration is an even more useful quantity in mechanics