Lecture 33: The Inertia Tensor

- We found last time that the kinetic energy and angular momentum of a rotating object were:

\[
T_{\text{rot}} = \frac{1}{2} \sum_{i,j} \omega_i \omega_j I_{ij} \quad \text{and} \quad L_i = \sum_j \omega_j I_{ij}
\]

where

\[
I_{ij} \equiv \sum_{\alpha} m_{\alpha} \left[ \delta_{ij} \left( \sum_k x_{\alpha,k}^2 \right) - x_{\alpha,i} x_{\alpha,j} \right]
\]

- So the six numbers represented by the \( I_{ij} \) tell us all we need to know about the rigid body to determine \( T \) and \( L \) for a given \( \omega \)
Inertia Tensor in Different Coordinate Systems

• So far I’ve called the numbers $I_{ij}$ a “tensor”, but have provided no definition of what a tensor is.

• To provide one, consider what happens to the $I_{ij}$ if one rotates the coordinate system.
  – We’ll start with the simpler case of observing how a vector transforms under such a rotation:

  • The components of $\mathbf{V}$ can be defined as $(x, y)$ or as $(x', y')$.

  • This implies a relationship between the variables.
For example, we can find $x'$ in terms of $x$ and $y$:

\[ x' = a + b \]

\[ a = \frac{x}{\cos \theta} \]

\[ b = c \sin \theta = (y - a \sin \theta) \sin \theta = y \sin \theta - a \sin^2 \theta \]

\[ x' = y \sin \theta - a \sin^2 \theta + a = y \sin \theta + a \left(1 - \sin^2 \theta\right) \]

\[ = y \sin \theta + a \cos^2 \theta = y \sin \theta + x \cos \theta \]
• Similarly, we find that:

\[ y' = -x \sin \theta + y \cos \theta \]

• We can rewrite these relations in terms of cosines only:

\[ x' = x \cos \theta + y \cos \left( \frac{\pi}{2} - \theta \right) \]

\[ y' = x \cos \left( \frac{\pi}{2} + \theta \right) + y \cos \theta \]

• Now let’s change notation slightly
  – Rather than \( x, y, z \) we’ll write \( x_1, x_2, x_3 \)
  – Define the cosine of the angle between the \( i \)th primed axis and the and \( j \)th unprimed axis as \( \lambda_{ij} \)
• With this notation, we can simplify the relation between the primed and unprimed coordinates:

\[ x_1' = x_1 \lambda_{11} + x_2 \lambda_{12} \]
\[ x_2' = x_1 \lambda_{21} + x_2 \lambda_{22} \]

• This can be easily extended to higher dimensions:

\[ x_1' = x_1 \lambda_{11} + x_2 \lambda_{12} + x_3 \lambda_{13} \]
\[ x_2' = x_1 \lambda_{21} + x_2 \lambda_{22} + x_3 \lambda_{23} \]
\[ x_3' = x_1 \lambda_{31} + x_2 \lambda_{32} + x_3 \lambda_{33} \]

• So everything we need to know about the rotation is contained in the \( \lambda \)s
• We can write the $\lambda_{ij}$ as a matrix:

$$\lambda = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{bmatrix}$$

The rotation matrix

• Note that rotations never change the length of a vector
• This implies that the elements of the rotation matrix have the following property:

$$\sum_k \lambda_{ik} \lambda_{mk} = \delta_{im}$$

• Matrices that behave this way are called “orthogonal”
To determine how I transforms under rotations, we start with some related quantities.

Since L is a vector, its components must transform according to:

\[ L'_k = \sum_m \lambda_{mk} L_m \]

By the same token:

\[ \omega'_l = \sum_j \lambda_{jl} \omega_j \]

And we also know that:

\[ L'_k = \sum_l I'_{kl} \omega'_l \]
• So we must have:

\[ \sum_{m} \lambda_{mk} L_m = \sum_{l} I'_{kl} \sum_{j} \lambda_{jl} \omega_j \]

\[ \sum_{k} \lambda_{ik} \sum_{m} \lambda_{mk} L_m = \sum_{k} \lambda_{ik} \sum_{l} I'_{kl} \sum_{j} \lambda_{jl} \omega_j \]

\[ \sum_{m,k} \lambda_{ik} \lambda_{mk} L_m = \sum_{j} \left( \sum_{k,l} \lambda_{ik} \lambda_{jl} I'_{kl} \right) \omega_j \]

• We learned before that the rotation matrix is orthogonal, so that:

\[ \sum_{k} \lambda_{ik} \lambda_{mk} = \delta_{im} \]

• Using this, the above equation becomes:

\[ \sum_{m,k} \delta_{im} L_m = \sum_{j} \left( \sum_{k,l} \lambda_{ik} \lambda_{jl} I'_{kl} \right) \omega_j \]

\[ L_i = \sum_{j} \left( \sum_{k,l} \lambda_{ik} \lambda_{jl} I'_{kl} \right) \omega_j \]
• But we also know that $L_i = \sum_j I_{ij} \omega_j$

• The only way it can all work out is for $I$ and $I'$ to have the following relationship:

$$I_{ij} = \sum_{k,l} \lambda_{ik} \lambda_{jl} I'_{kl}$$

• Any collection of numbers that follow this rule for rotations is called a “tensor”
  – The number of indices determines the rank of the tensor
  – Moment of inertia is a second-rank tensor

• In general, a tensor of arbitrary rank will transform according to:

$$T_{abcd...} = \sum_{i,j,k,l...} \lambda_{ai} \lambda_{bj} \lambda_{ck} \lambda_{dl} ... T'_{ijkl...}$$
• This means that a first-rank tensor transforms as:

\[ T_a = \sum_i \lambda_{ai} T'_i \]

In other words, it’s a vector!

• For second-rank tensors, the transformation is the same as matrix multiplication:

\[ I = \lambda_1 \lambda^t \]

• Another property of orthogonal matrices is that their transpose is equal to their inverse. Using this, we can write:

\[ I = \lambda_1 \lambda^{-1} \]

• Any transformation of this type is called a similarity transformation.
More On Angular Momentum

• Returning to the definition of $L_i$,

$$L_i = \sum_j I_{ij} \omega_j$$

we can multiply this by $\frac{1}{2} \omega_i$ and sum over $i$:

$$\frac{1}{2} \sum_i \omega_i L_i = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j = T_{rot}$$

• In other words:

$$T_{rot} = \frac{1}{2} \omega \cdot L = \frac{1}{2} \omega \cdot I \cdot \omega$$

• So we see that a tensor times a vector gives a vector, while a tensor times two vectors gives a scalar