Lecture 6: Phase Space and Damped Oscillations
Oscillations in Multiple Dimensions

• The previous discussion was fine for oscillation in a single dimension

• In general, though, we want to deal with the situation where:
  \[ \mathbf{F} = -k \mathbf{r} \]

• No big deal – we can consider one component at a time:
  \[ F_{x_1} = m \ddot{x}_1 = -kx_1 \]
  \[ F_{x_2} = m \ddot{x}_2 = -kx_2 \]
  \[ \vdots \]

• Solutions are:
  \[ x_1(t) = A_1 \sin(\omega_o t + \delta_1) \]
  \[ x_2(t) = A_1 \sin(\omega_o t + \delta_2) \]
  \[ \vdots \]

Note that frequency is the same in each direction

So we just need to solve the same equation as before, once for each dimension
The Two-Dimensional Case

- We can find the path taken by a particle undergoing two-dimensional oscillation
  - Easiest if we start with a little trick – rewrite $x_2$ as:

$$x_2 = A_2 \sin \left( \omega_o t + \delta_1 - \delta_1 + \delta_2 \right)$$

$$= A_2 \left[ \sin (\omega_o t + \delta_1) \cos (\delta_2 - \delta_1) + \cos (\omega_o t + \delta_1) \sin (\delta_2 - \delta_1) \right]$$

$$= A_2 \left[ \frac{x_1}{A_1} \cos (\delta_2 - \delta_1) + \sqrt{1 - \left( \frac{x_1}{A_1} \right)^2} \sin (\delta_2 - \delta_1) \right]$$

$$A_1 x_2 - A_2 x_1 \cos (\delta_1 - \delta_2) = A_2 \sqrt{A_1^2 - x_1^2} \sin (\delta_2 - \delta_1)$$

$$A_1^2 x_2^2 - 2 A_1 A_2 x_1 x_2 \cos (\delta_1 - \delta_2) + A_2^2 x_1^2 \cos^2 (\delta_1 - \delta_2)$$

$$= A_1^2 A_2^2 \sin^2 (\delta_2 - \delta_1) - A_2^2 x_1^2 \sin^2 (\delta_2 - \delta_1)$$
• Finally, we have:

\[ A_1^2 x_2^2 - 2A_1A_2 x_1 x_2 \cos(\delta_1 - \delta_2) + A_2^2 x_1^2 = A_1^2 A_2^2 \sin^2 (\delta_2 - \delta_1) \]

- This is the equation for an ellipse

- Note that the shape of the ellipse depends only on the relative amplitudes and difference in phase between the motion in each direction

\[ \delta_2 - \delta_1 = 0 \]

\[ \delta_2 - \delta_1 = 30^\circ \]

\[ \delta_2 - \delta_1 = 90^\circ \]

\[ \delta_2 - \delta_1 = 120^\circ \]

\[ \delta_2 - \delta_1 = 180^\circ \]
Lissajous Curves

- The two-dimensional motion becomes even more interesting if the frequencies are different in each dimension.
- As long as the ratio of frequencies is a rational number, the motion is still periodic:

\[ \omega_2 = \frac{7}{5} \omega_1 \]
\[ \delta_2 - \delta_2 = 90' \]

Note that there are 7 maxima in \( x_2 \) and 5 maxima in \( x_1 \).
Phase Space

• Thinking a bit more abstractly about the solution of the one-dimensional oscillator, we recall that two initial conditions needed to be specified
  – A common occurrence in mechanics, since the equation of motion is a second-order differential equation

• We can take the initial conditions to be specified by an initial $x$ and $\dot{x}$
  – i.e., once we know these two quantities at a given point in time, we can determine what their values will be at any other time

• We can represent this information graphically in “phase space”
  – let one axis be $x$, and the other be $\dot{x}$
• The motion in phase space can be determined from the known functions $x(t)$ and $\dot{x}(t)$ or directly from the equation of motion:

\[
\frac{dx}{dt} = -\omega_0^2 x \\
\frac{d\dot{x}}{dt} = -\omega_0^2 \frac{x}{\dot{x}} \\
\dot{x}d\dot{x} = -\omega_0^2 xdx \\
\dot{x}^2 + \omega_0^2 x^2 = C = A^2 \omega_0^2
\]

A similar procedure can be used for other types of motion – sometimes easier than integrating to find $x$. This is an ellipse.
Phase Space Ellipses

- The shape of the ellipse is determined by $\omega_o$, and the size by $A$:
  - $\omega_o = 0.5$

  $A = 100$

  
  ![Ellipse A=100, $\omega_o=0.5$](image1)

  $A = 50$

  ![Ellipse A=50, $\omega_o=0.5$](image2)

  $A = 10$

  ![Ellipse A=10, $\omega_o=0.5$](image3)

  - $\omega_o = 0.2$

  $A = 100$

  ![Ellipse A=100, $\omega_o=0.2$](image4)

  $A = 50$

  ![Ellipse A=50, $\omega_o=0.2$](image5)

  $A = 10$

  ![Ellipse A=10, $\omega_o=0.2$](image6)
Features of Phase Space

• A particle’s motion is represented by a curve in phase space
  – Periodic motion is represented by a closed curve, like the ellipses in the previous example

• No two curves in phase space can cross
  – Or else the particle would have two possible futures for a given $x$ and $\dot{x}$

• Motion in two and three dimensions can also be treated in phase space
  – For the two-dimensional case, phase space is four-dimensional
  – Axes are $x_1, x_2, \dot{x}_1, \dot{x}_2$
  – In general, phase space has $2N$ dimensions for motion in $N$ “real” spatial dimensions
Damped Oscillations

• Almost all real oscillators experience some resistance to their motion
  – In general, such resistance is called “damping”

• As with the resistive forces studied earlier, the precise form of the damping can vary

• But we can explore many of the features of damping by assuming the force is proportional to velocity
  – In this case, the equation of motion becomes (in one dimension):

\[
m \ddot{x} = -kx - b\dot{x}
\]

\[
\dddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0
\]
• This type of differential equation is called a “linear, homogenous equation”. Assume the solution is of the form $x = e^{rt}$:

$$r^2 e^{rt} + 2\beta r e^{rt} + \omega_o^2 e^{rt} = 0$$
$$r^2 + 2\beta r + \omega_o^2 = 0$$
$$r = \frac{-2\beta \pm \sqrt{(2\beta)^2 - 4\omega_o^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$$

• Note that either of these solutions for $r$ gives an acceptable solution to the equation
  – Also, multiplying the solution by a constant results in an acceptable solution
• Therefore, the general solution is the following:

\[ x(t) = e^{-\beta t} \left[ A_1 e^{\sqrt{\beta^2 - \omega^2}t} + A_2 e^{-\sqrt{\beta^2 - \omega^2}t} \right] \]

• However, the qualitative nature of the motion depends on the relative sizes of \( \beta \) and \( \omega_o \)

• Case 1: \( \beta < \omega_o \)
  – The quantity \( \sqrt{\beta^2 - \omega^2} \) is imaginary
  
  – We can write it as:

\[
\sqrt{\beta^2 - \omega^2} = i\sqrt{\omega^2 - \beta^2} \equiv i\omega_1
\]

\[ x(t) = e^{-\beta t} \left[ A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t} \right] \]

\[ = e^{-\beta t} \left[ A_1 \left( \cos \omega_1 t + i \sin \omega_1 \right) + A_2 \left( \cos \omega_1 t - i \sin \omega_1 \right) \right] \]
\[ x(t) = e^{-\beta t} \left[ (A_1 + A_2) \cos \omega_1 t + i(A_1 - A_2) \sin \omega_1 t \right] \]

- \(A_1\) and \(A_2\) are complex numbers, but our answer must be real
  - Implies that \(A_1\) and \(A_2\) are complex conjugates
  - Can write them as: \(A_1 = Ae^{i\delta}\), \(A_2 = Ae^{-i\delta}\)

- We now have:
  \[
  x(t) = e^{-\beta t} \left[ A(\cos \delta + i \sin \delta + \cos \delta - i \sin \delta) \cos \omega_1 t \\
  + iA(\cos \delta + i \sin \delta - \cos \delta + i \sin \delta) \sin \omega_1 t \right] \\
  = Ae^{-\beta t} \left[ 2 \cos \delta \cos \omega_1 t - 2 \sin \delta \sin \omega_1 t \right] \\
  = 2Ae^{-\beta t} \cos(\omega_1 t + \delta)
  \]

- Since we can always redefine the constant \(A\) to get rid of the 2 in front of the equation, the general solution is:
  \[ x(t) = Ae^{-\beta t} \cos(\omega_1 t + \delta) \]
Properties of underdamped motion

- An underdamped system still oscillates:

- The quantity $\omega_1$ can still be related to the time interval between crossings of the $x$ axis.
- For light damping, $\omega_1$ is very close to $\omega_0$.

Note, though, that the motion is not periodic – it never returns to the same point with the same velocity as before.
Underdamped Motion in Phase Space

- Since the motion is not periodic, we no longer get closed loops. In addition to amplitude, path depends on $\beta$:

$\omega_0 = 0.5, \ A = 100$

$\beta = 0.05$

$\beta = 0.1$

$\beta = 0.25$